Flag algebra method in extremal combinatorics

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Quasirandomness

- When does a graph look as $G_{n,1/2}$?
- seminal paper by Chung, Graham, Wilson (1989)
  the density of every subgraph is as in $G_{n,1/2}$
  this is equivalent to the density of K_2 and C_4
  as well as to many other statements
- Is the same true for permutations?
  yes, the densities of 4-point subpermutations
  subpermutation: 453216 $\longrightarrow$ 213

Flag algebras

- flag algebra framework [Razborov, 2010]
  graph limits [Lovász, Szegedy, 2006] and
  [Borgs, Chayes, Lovász, Sós, Vesztergombi, 2008]
- $p(\sigma, \Pi)$ is the probability that random $|\sigma|$ points of Π induce σ
- consider a sequence $\Pi_1, \Pi_2, \ldots$
- there always exists a subsequence such that for every σ
  the sequence $p(\sigma, \Pi_1), p(\sigma, \Pi_2), \ldots$ converges
- the limit object is a function q from permutations to reals
  in fact, q yields a homomorphism from an algebra of
  formal linear combinations of permutations to reals

Basic properties

- linear extension:

  $q\left(\alpha\sigma + \alpha'\sigma'\right) := \alpha q(\sigma) + \alpha' q(\sigma')$

- lifting up:

  $q\left(12\right) = q\left(\frac{3}{3}123 + \frac{2}{3}132 + \frac{2}{3}213 + \frac{1}{3}231 + \frac{1}{3}312\right)$

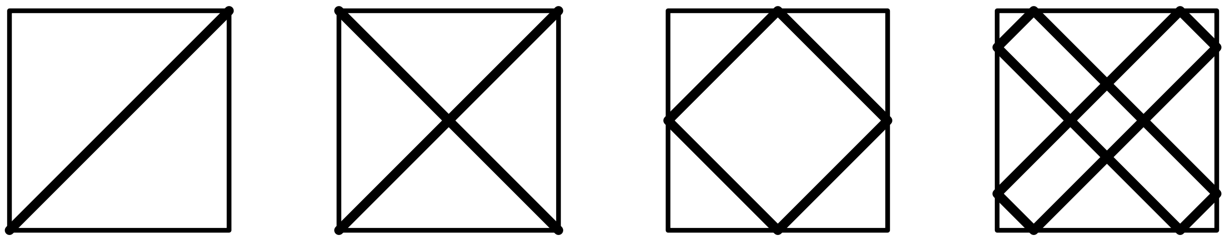
- product:

  $q\left(12\right) \times q\left(21\right) = q\left(\frac{1}{6}1243 + \frac{1}{6}1324 + \frac{2}{6}1342 + \frac{2}{6}1423 + \frac{3}{6}1432 + \cdots\right)$

  $12 \times 21 := \frac{1}{6}1243 + \frac{1}{6}1324 + \frac{2}{6}1342 + \frac{2}{6}1423 + \frac{3}{6}1432 + \cdots$

Limit object

- Can we describe the limit in analytic terms?
- for graphs: a measurable function $[0,1]^2 \to [0,1]$
  for limits: probabilistic measure on $[0,1]^2$ with unit marginals
  $\mu_q([a,b] \times [0,1]) = \mu_q([0,1] \times [a,b]) = b - a$
- similar approach by Hoppen et al.



Rooted homomorphisms

- consider a "large" permutation Π consistent with q
- fix a set S of points
  consider subpermutations induced by supersets of S
  we get a mapping q^S depending on the choice of S
- random distribution of such mappings q^S
- averaging argument: $q\,(12) = \mathbb{E}_1 q^1(12 + 12)$
- homomorphism q uniquely determines corresponding
  random distributions for all choices of roots S
- possible to define multiplication

Relation to the limit

- a close relation between the limit measure and q
  in our case: correct density of 4-point permutations but $\mu_q \neq \mu_u$
- suppose $f(x,y) = q^1(\sigma)$ if $1 = (x,y)$
  $\int f(x,y)\,\mathrm{d}\mu_q = \mathbb{E}_1 q^1(\sigma) = q\left(\llbracket\sigma\rrbracket_1\right)$
- example: $f(x,y) = x = q^1(12 + 21)$
  $\int x^2\,\mathrm{d}\mu_q = q\left(\llbracket(12+21)\times(12+21)\rrbracket_1\right)$
- Can we integrate based on the uniform measure μ_u?
  Yes! Choose two points according to μ_u.
  one determines x and the other y

The proof

$$\frac{1}{81} = \left(\int \alpha(x,y)xy \, \mathrm{d}\mu_q\right)^2 \le \left(\int \alpha(x,y)^2 \, \mathrm{d}\mu_q\right) \cdot \left(\int x^2y^2 \, \mathrm{d}\mu_q\right)$$

$$= \frac{1}{9}\left(4 \cdot \int_{[0,1]^2} \alpha(x,y)xy \, \mathrm{d}\mu_u - \int (1 - x^2 - y^2) \, \mathrm{d}\mu_q\right)$$

$$= \frac{1}{9}\left(4 \cdot \int \alpha(x,y)xy \, \mathrm{d}\mu_u - \frac{1}{3}\right)$$

$$\le \frac{4}{9}\sqrt{\int \alpha(x,y)^2 \, \mathrm{d}\mu_u}\sqrt{\int x^2y^2 \, \mathrm{d}\mu_u} - \frac{1}{27} = \frac{1}{81}$$

μ_q is the limit and $\alpha(x,y) = \mu_q([0,x] \times [0,y])$

Thank you for your attention!