Subcube isoperimetry and power of coalitions

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Isoperimetric problems

The notion of isoperimetry

For the area A of the planar region enclosed by a curve of length L it holds $4\pi A \leq L^2$, with equality if and only if the curve is a circle.

The edge isoperimetric parameter

$$\Phi_E(G,k) = \min_{S \subset V} \{|E(S,\overline{S})|;\ |S| = k\}$$

The expansion

$$h(G) = \min_{k \leq |V|/2} \frac{\Phi_E(G,k)}{k}$$

The problems of determining these parameters for general G are co-NP hard.

Spectral methods

For a d-regular G with the second eigenvalue λ_2 of its adjacency matrix,

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}$$

The edge isoperimetric problem in the hypercube

Let $f_n(k) = \max_{S \subset V}\{|E(Q_n[S])|;\ |S| = k\}$. That is, $\Phi_E(Q_n, k) = nk - 2f_n(k)$.

Theorem [Harper; Bernstein; Hart]

$$f_n(k) = \sum_{i=0}^{k-1} h(i)$$

where $h(i)$ is the number of 1's in the binary representation of i.

Extremal sets

A set $S \subset \{0, 1\}^n$ is good if $|S| = 1$ or there are $C_m \simeq Q_m$, $C_{m+1} \simeq Q_{m+1}$, $2^m < |S| \leq 2^{m+1}$ s.t. $V(C_m) \subset S \subseteq V(C_{m+1})$ and $S \setminus V(C_m)$ is good.

good sets (up to isomorphism) $\approx$ initial segments in the lexicographical order

A useful estimate [Chung, Füredi, Graham, Seymour]

$$\Phi_E(Q_n, k) \geq k(n - \log_2 k)$$

with equality for $k = 2^d$ attained by a d-dimensional subcube.

Subcube isoperimetric problem in the hypercube

Let $f_n(k,d) = \max_{S \subset V}\{\#_d(S);\ |S| = k\}$ where $\#_d(S)$ denotes the number of (induced) subcubes of dimension d in $Q_n[S]$. (inner subcubes)

Theorem

$$f_n(k,d) = \sum_{i=0}^{k-1} \binom{h(i)}{d}$$

for every $k > 0$, $d \geq 0$ and the maximum is attained by all good sets of size k.

Remark: good sets are optimal for every $d \geq 0$.

Let $g_n(k,d) = \min_{S \subset V}\{\sigma_d(S);\ |S| = k\}$ where $\sigma_d(S)$ denotes the number of (induced) Q_d's with a vertex in S and a vertex in $\overline{S}$. (border subcubes)

Corollary

$$g_n(k,d) = \binom{n}{d} 2^{n-d} - f_n(k,d) - f_n(2^n - k, d)$$

for every $n \geq 1$, $0 < k < 2^n$, $d \geq 0$.

Labeling of the hypercube

For a bijection $c : \{0,1\}^n \to [0, 2^n - 1]$, a set $S \subseteq \{0,1\}^n$, and $d \geq 1$ let

$$\delta_c(S) = |S| \max_{u \in S} c(u) - \sum_{u \in S} c(u) \qquad \text{(the maximal deviation of } c \text{ on } S\text{)},$$

$$\Delta_n^d(c) = \sum_{Q_d \simeq C \subseteq Q_n} \delta_c(V(C)) \qquad \text{(the total maximal deviation of } c \text{ on } Q_d\text{'s)}.$$

Question: Which labeling c of $V(Q_n)$ minimizes $\Delta_n^d(c)$ for given $n \geq d \geq 1$?

Integer coding scenario

1. encode (uniformly) chosen $0 \leq I < 2^n$ by $u = c^{-1}(I) \in \{0,1\}^n$,
2. (at most) d coordinates D are chosen uniformly in random,
3. an adversary with knowledge of c may flip any bit from D in u => u',
4. decode $I' = c(u')$.

Problem: Find coding c that minimizes expected error $I' - I$.

Subcube isoperimetry and total max. deviation

$\sigma_d(S)$ counts each border subcube once. How much border subcubes hit S?

The relevance of a set $S \subseteq \{0,1\}^n$ in border subcubes of dimension d is

$$\rho_d(S) = \sum_{Q_d \simeq C \not\subseteq Q_n[S]} |V(C) \cap S| = \binom{n}{d}|S| - 2^d \#_d(S).$$

For a bijection $c : \{0,1\}^n \to [0, 2^n - 1]$ and $1 \leq l \leq 2^n$ let

$$\Theta_n^d(c,l) = \rho_d(\{c^{-1}(0), \ldots, c^{-1}(l-1)\}).$$

Lemma

$$\Delta_n^d(c) = \sum_{l=1}^{2^n} \Theta_n^d(c,l)$$

Theorem

The binary coding c minimizes $\Delta_n^d(c)$ for every $d \geq 1$.

Optimal labelings - open questions

1) Maximize total maximal deviation.

   Question: Which labelings c of $V(Q_n)$ have maximal $\Delta_n^d(c)$?

2) Minimize largest maximal deviation.

   Question: Which labelings c of $V(Q_n)$ minimize $\max_{Q_d \simeq C \subseteq Q_n} \delta_c(V(C))$?

   Both questions seem to be open even for $d = 1$.

3) Generalization to (uniform) hypergraphs $\mathcal{H}$.

   Question: Which labelings c of $V(\mathcal{H})$ minimize $\Delta_{\mathcal{H}}(c) = \sum_{H \in E(\mathcal{H})} \delta_c(H)$?

   The hyperedge isoperimetry & relevance approach requires an order on $V(\mathcal{H})$ whose initial segments minimize relevance in border hyperedges.

4) Question: Which (classes of) hypergraphs have such an order?

Influence in simple voting games

We have n players with 0/1 votes, an outcome function $f : \{0,1\}^n \to \{0,1\}$.

What is the probability that the player $i \in [n]$ can influence the result?

Influence (Banzhaf power index)

$$I_f(i) = \Pr_x[f(x) \neq f(x \oplus e_i)], \quad I_f = \sum_{i \in [n]} I_f(i)$$

Smallest total influence [Hart]

$$\min_{f : p_1(f) = k/2^n} I_f = \frac{\Phi_E(Q_n, k)}{2^{n-1}} \quad \text{where } p_1(f) = \Pr_x[f(x) = 1] \ \text{(bias)}$$

Theorem [Kahn, Kalai, Linial]

For every $f : \{0,1\}^n \to \{0,1\}$ with $p_1(f) = \frac{1}{2}$ there exists $i \in [n]$ with

$$I_f(i) \geq \frac{c \log n}{n} \qquad \text{where } c \text{ is an absolute constant.}$$

Harmonic analysis of Boolean functions

A Boolean function: $f : \{-1, 1\}^n \to \{-1, 1\}$.

Fourier basis

$\{\chi_S\}_{S \subseteq [n]}$ in $\mathbb{R}^{\{-1,1\}^n}$ where $\chi_S(x) = \prod_{i \in S} x_i, \ \chi_\emptyset(x) = 1$ (characters)

Fourier transform

$$f = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S$$

Inner product and induced norm

$$\langle f, g \rangle = \mathbb{E}_x[f(x)g(x)], \quad \|f\|_2 = \sqrt{\langle f, f \rangle}$$

Since $\{\chi_S\}_{S \subseteq [n]}$ orthonormal,

$$\widehat{f}(S) = \langle f, \chi_S \rangle, \quad \langle f, g \rangle = \sum_{S \subseteq [n]} \widehat{f}(S) \widehat{g}(S),$$

$$\mathbb{E}_x[f(x)] = \langle f, \chi_\emptyset \rangle = \widehat{f}(\emptyset), \quad 1 = \|f\|^2 = \sum_{S \subseteq [n]} \widehat{f}^2(S).$$

Influence in Fourier coefficients

$$I_f(i) = \mathbb{E}_x[\mathbb{V}_{x_i}[f(x)]] = \sum_{S : i \in S} \widehat{f}^2(S), \quad I_f = \sum_{S \subseteq [n]} |S| \widehat{f}^2(S)$$

Influence of coalitions

$$I_f(S) = \mathrm{Pr}_{x\setminus S}[\mathbb{E}_S^2[f(x)] < 1], \quad I_f^d = \sum_{\substack{S\subseteq[n] \\ |S|=d}} I_f(S)$$

Smallest total coalitional influence

$$\min_{f:p_1(f)=k/2^n} I_f^d = \frac{g_n(k,d)}{2^{n-d}}$$

Lemma [Ben-Or, Linial]

For every Boolean function f there is a monotonous g such that

$$p_1(g) = p_1(f) \quad \text{and} \quad I_g(S) \leq I_f(S) \quad \text{for every } S \subseteq [n].$$

Influence for monotonous functions

$$I_f(S) = \sum_{\substack{T\subseteq S \\ |T| \text{ odd}}} \widehat{f}(T), \quad I_f^d = \sum_{\substack{S\subseteq[n] \\ |S| \text{ odd}}} \widehat{f}(S)\binom{n-|S|}{n-d}$$

Harmonic analysis of good functions

$f : \{-1,1\}^n \to \{-1,1\}$ s.t. $f^{-1}(1)$ is a good set of size $k = \sum_{i=1}^n b_i 2^{n-i} < 2^n$

Representation of f by formula φ_1 from k

$$\varphi_n : \begin{cases} \top & b_n = 0 \\ x_n & b_n = 1 \end{cases}, \qquad \varphi_i : \begin{cases} x_i \vee (\varphi_{i+1}) & b_i = 0 \\ x_i \wedge (\varphi_{i+1}) & b_i = 1 \end{cases}$$

Fourier transform from formula

$$\mathbf{p}_{x_i} = x_i \qquad \mathbf{p}_{\varphi \wedge \psi} = \mathbf{p}_\varphi \mathbf{p}_\psi$$

$$\mathbf{p}_{\neg\varphi} = -\mathbf{p}_\varphi \qquad \mathbf{p}_{\varphi \vee \psi} = \mathbf{p}_\varphi + \mathbf{p}_\psi - \mathbf{p}_\varphi \mathbf{p}_\psi$$

Fourier coefficients from k

$$\widehat{f}(S) = \left(\frac{c_j}{2} - \frac{1}{2^n} - \sum_{l=j+1}^{n} \frac{c_l}{2^l} \right) \prod_{i \in S} c_i \quad \text{where } j = \max(S), c_i = 1 - 2b_i \in \{-1,1\}$$

$$I_f = \sum_{i \in [n]} \widehat{f}(\{i\}) = 1 - \frac{1}{2^n} - \frac{1}{2^n} \sum_{i=1}^{n} c_i - \sum_{i<j} \frac{c_i c_j}{2^j}$$

Open problems

1) An alternative (straightforward) proof of the (exact) subcube isoperimetry through harmonic analysis.
2) An existence of highly influential coalitions - symmetry breaking (improvements in known results).
3) Fibonacci isoperimetry - players in coalitions cannot consecutively vote 1.
4) An (exact) subcube isoperimetry in Hamming graphs, ...
5) Connections between (minimal) representations of Boolean functions and influence of coalitions.

The story of Dido

Dido - Queen of Carthage / Vergilius: Aeneid

P.-N. Guérin: Aeneas tells Dido the misfortunes of the Trojan city.