Classes of Sparse Combinatorial Objects

From Structure to Algorithms

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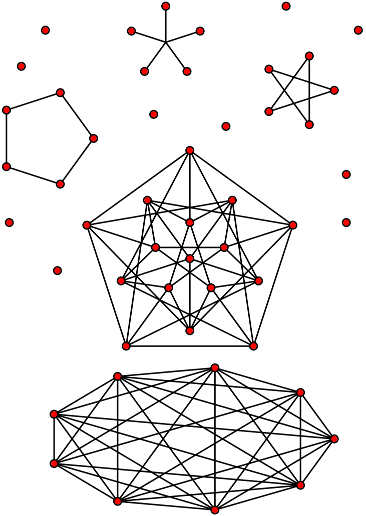
October 13-16 2011, Beroun

Classification

What is a sparse graph?



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- subgraphs, minors, homomorphic images?

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- K_t? $\Omega(n^2)$ edges? $\Omega(n^{10})$ copies of ?
- high chromatic number? large minimum degree?

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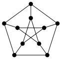
- K_t? $\Omega(n^2)$ edges? $\Omega(n^{10})$ copies of  ?
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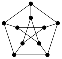
- K_t? $\Omega(n^2)$ edges? $\Omega(n^{10})$ copies of  ?
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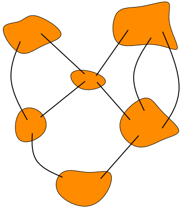
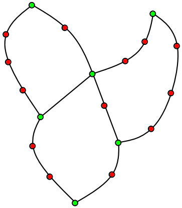
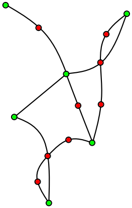
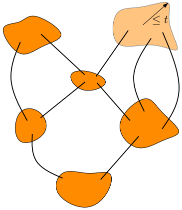
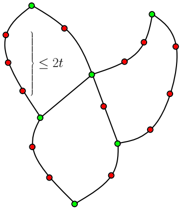
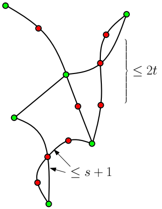
Every kind of minors . . .

Minor	Topological minor	Immersion
		
$\delta > ct\sqrt{\log t} \Rightarrow K_t$	$\delta > ct^2 \Rightarrow K_t$	$\delta > ct \Rightarrow K_t$
Kostochka, Thomason	Komlós and Szemerédi, Bollobás and Thomason	DeVos et al.

Every kind of shallow minors . . .

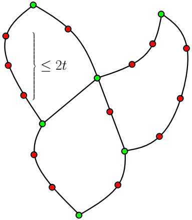
Shallow Minor	Shallow Topological minor	Shallow Immersion
		

$$G \nabla t \quad \supseteq \quad G \widetilde{\nabla} t \quad \subseteq \quad G \overset{\propto}{\nabla} (t, s+1)$$

Topological resolution of a class $\mathscr{C}$

Shallow topological minors at depth t:

$$\mathscr{C}\,\widetilde{\nabla}\,t = \{H : \text{ some } \leq 2t\text{-subdivision of H is present in some } G \in \mathscr{C}\}.$$



Example: $\mathscr{C}\,\widetilde{\nabla}\,0$ is the monotone closure of $\mathscr{C}$.

Topological resolution in time:

$$\mathscr{C} \subseteq \mathscr{C}\,\widetilde{\nabla}\,0 \subseteq \mathscr{C}\,\widetilde{\nabla}\,1 \subseteq \ldots \subseteq \mathscr{C}\,\widetilde{\nabla}\,t \subseteq \ldots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\infty$$

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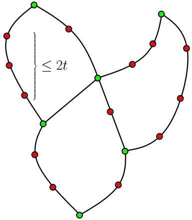

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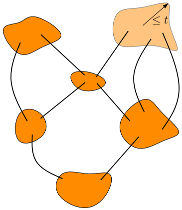
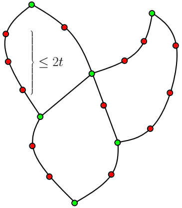
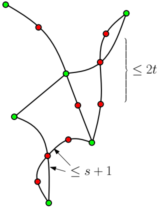
Every kind of resolutions . . .

Minor resolution	Topological resolution	Immersion resolution
		
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Taxonomy of Classes

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Theorem (Nešetřil, POM, 2010)

Same classification if ∇ or $\stackrel{\infty}{\nabla}$ instead of $\widetilde{\nabla}$.

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- Class of G such that $\Delta(G) \leq f(\mathrm{girth}(G))$
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- Random graphs $G(n, d/n)$
  $\exists$ bounded expansion class $\mathscr{R}_d$ s.t. $G(n, d/n) \in \mathscr{R}_d$ a.a.s.

ω-expansion and vertex separators

Theorem (Plotkin, Rao, Smith; 1994 — Wulff-Nilsen; 2011)

For integers l, h and a graph G of order n:

- *either $\omega(G \,\triangledown\, (l \log n)) \geq h$,*
- *or G has a vertex separator of size at most $O(n/l + lh^2 \log n)$*

Theorem (Nešetřil, POM)

If $\mathscr{C}$ is a monotone class such that

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Extremal logarithmic density of edges

Theorem (Jiang, 2010)

$$\mathrm{ex}(n, K_t^{(\leq p)}) = O(n^{1+\frac{10}{p}}).$$

$$\mathscr{C} \subseteq \mathscr{C}\,\widetilde{\nabla}\,0 \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,t \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\frac{10t}{\varepsilon} \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\infty$$

$$\|G\| > C_n\,|G|^{1+\varepsilon} \qquad\qquad K_n$$

$\|G\|=$ number of edges $\qquad |G|=$ number of vertices

Hence:

$$\limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,t} \frac{\log\|G\|}{\log|G|} > 1+\varepsilon \implies \limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,\frac{10t}{\varepsilon}} \frac{\log\|G\|}{\log|G|} = 2.$$

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Classification by logarithmic density of edges

Theorem (Class trichotomy — Nešetřil, POM, 2010)

Let $\mathscr{C}$ be an infinite class of graphs. Then

$$\sup_t \limsup_{G \in \mathscr{C} \,\widetilde{\nabla}\, t} \frac{\log \|G\|}{\log |G|} \in \{-\infty, 0, 1, 2\}.$$

- bounded size *class* $\iff$ $-\infty$ *or* 0*;*
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Classification by logarithmic density of anything

Theorem (Counting dichotomy; Nešetřil, POM, 2011)

Let $\mathscr{C}$ be an infinite class of graphs and let F be a graph with at least one edge. Then

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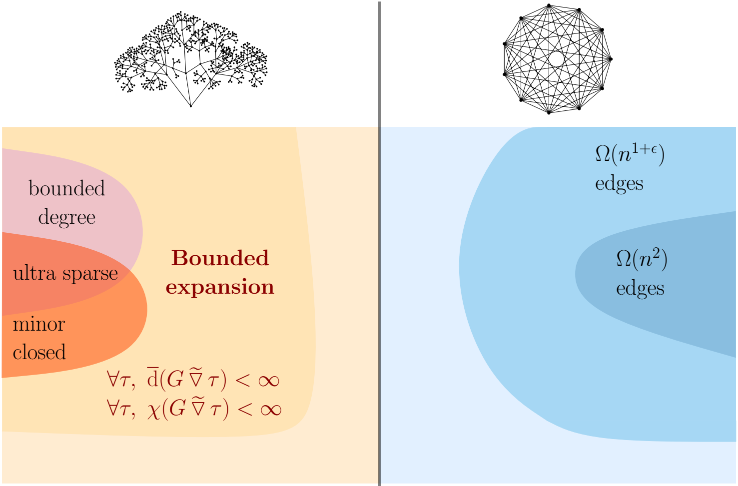
General diagram



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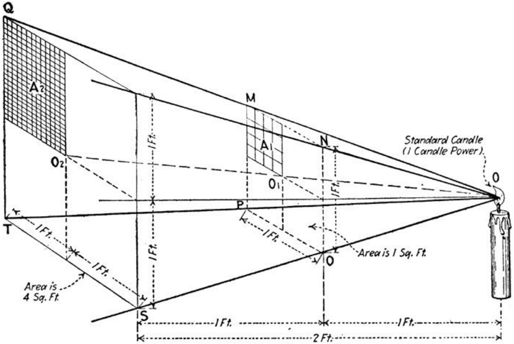
Grads (density vs depth)



grad and top-grad

The *greatest reduced average density (grad)* with rank r of a graph G is

$$\nabla_r(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \,\triangledown\, r\right\}$$

The *top-grad* with rank r of G is

$$\widetilde{\nabla}_r(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \,\widetilde{\triangledown}\, r\right\}$$

The *imm-grad* of rank (r, s) of G is

$$\overset{\propto}{\nabla}_{r,s}(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \,\overset{\propto}{\triangledown}\, (r, s)\right\}.$$

grad and top-grad

Theorem (Dvořák, 2007)

Let $r, d \geq 1$ be integers and let $p = 4(4d)^{(r+1)^2}$. If $\nabla_r(G) \geq p$, then G contains a subgraph F' that is a $\leq 2r$-subdivision of a graph F with minimum degree d.
Hence:

$$\widetilde{\nabla}_r(G) \leq \nabla_r(G) \leq 4(4\widetilde{\nabla}_r(G))^{(r+1)^2}$$

Theorem (Nešetřil, POM)

$$\widetilde{\nabla}_s(G \,\widetilde{\triangledown}\, r) \leq \widetilde{\nabla}_s(G \triangledown r) \leq 2^{r+2} 3^{(r+1)(r+2)} \widetilde{\nabla}_s(G \,\widetilde{\triangledown}\, r)^{(r+1)^2}.$$

Notice that $\widetilde{\nabla}_0(G \,\widetilde{\triangledown}\, r) = \widetilde{\nabla}_r(G)$ and $\widetilde{\nabla}_0(G \triangledown r) = \nabla_r(G)$.

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Lexicographic product and imm-grad

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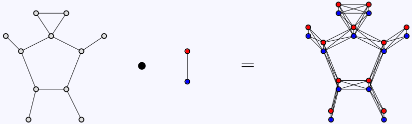

Theorem (Nešetřil, POM)

$$\widetilde{\nabla}_r(G \bullet K_p) \leq \max(2r(p-1)+1, p^2)\widetilde{\nabla}_r(G) + p - 1$$

Corollary

As

$$G \,\widetilde{\nabla}\, r \subseteq G \,\overset{\infty}{\nabla}\, (r,s) \subseteq (G \bullet \overline{K_s}) \,\widetilde{\nabla}\, r$$

all of $\nabla_r, \widetilde{\nabla}_r$ *and* $\overset{\infty}{\nabla}_{r,r+1}$ *are polynomially equivalent.*

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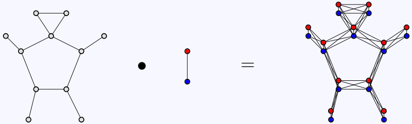

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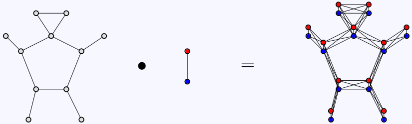

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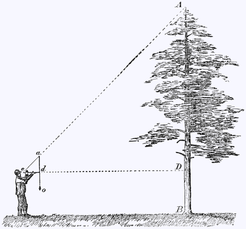
Trees

Tree-depth

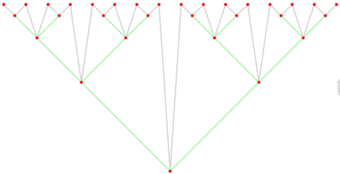
Definition



The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \mathrm{Closure}(Y).$$

(extends to infinite graphs ▸)



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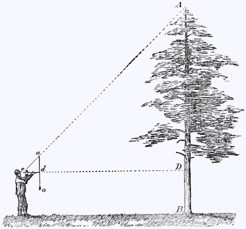

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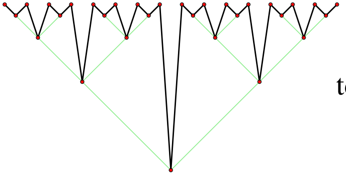

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Properties

- the tree-depth is minor-monotone:

$$H \text{ minor of } G \implies \mathrm{td}(H) \le \mathrm{td}(G).$$

- for every graph G it holds

$$\mathrm{tw}(G) \le \mathrm{pw}(G) \le \mathrm{td}(G) \le (\mathrm{tw}(G)+1)\log_2 |G|.$$

- there exists $F : \mathbb{N} \to \mathbb{N}$ such that every graph G of order greater than $F(\mathrm{td}(G))$ has a non-trivial involutive automorphism.

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- there exists $F : \mathbb{N} \to \mathbb{N}$ such that every graph G of order greater than $F(\mathrm{td}(G))$ has a non-trivial involutive automorphism.

Properties

- the tree-depth is minor-monotone:

$$H \text{ minor of } G \quad \Longrightarrow \quad \mathrm{td}(H) \leq \mathrm{td}(G).$$

- for every graph G it holds

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Further properties

Theorem (Nešetřil, POM)

For a monotone class of graphs, the following conditions are equivalent:

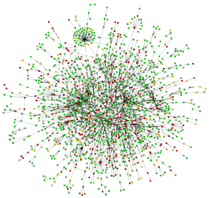
- *graphs in $\mathscr{C}$ have sublinear vertex separator,*
- *graphs in $\mathscr{C}$ have sublinear tree-width,*
- *graphs in $\mathscr{C}$ have sublinear path-width,*
- *graphs in $\mathscr{C}$ have sublinear tree-depth.*

Tree-depth of random graphs

Theorem (Perarnau, Serra, 2011)

Let $G \in \mathscr{G}(n,p)$.

- *If $p = \omega(n^{-1})$ then a.a.s.* $\mathrm{td}(G) = n - o(n)$
- *If $p = c/n$ with $c > 0$:*
  - *if $c < 1$, then a.a.s.* $\mathrm{td}(G) = \Theta(\log\log n)$;
  - *if $c = 1$, then a.a.s.* $\mathrm{td}(G) = \Theta(\log n)$;
  - *if $c > 1$, then a.a.s.* $\mathrm{td}(G) = \Theta(n)$.



First-order definition

Theorem (Ding, 1992 — Nešetřil, POM)

The poset of the graphs with tree depth at most t ordered by induced subgraph inclusion $\subseteq_i$ is a well quasi-order.

Corollary (First-order definition)

For every integer t, there exists a first-order formula τ_t such that for every graph G it holds

$$\mathrm{td}(G) \leq t \qquad \Longleftrightarrow \qquad G \models \tau_t.$$

Tree-depth of countable graphs

At most countable graphs G and H are elementarily equivalent if they satisfy the same first-order properties. This is denoted by $G \equiv H$.

For $\mathfrak{G}$ and $\mathfrak{H}$ equivalence classes of graphs for $\equiv$, define the ultrametric

$$\operatorname{dist}(\mathfrak{G}, \mathfrak{H}) = 2^{-\sup\{n,\ G \equiv^n H, \quad G \in \mathfrak{G}, H \in \mathfrak{H}\}}.$$

Theorem

Let $t \in \mathbb{N}$. Define

$$\mathscr{T}_t = \{G \text{ finite} : \operatorname{td}(G) \leq t\},$$
$$\mathscr{T}_t^\star = \{G \text{ at most countable} : \operatorname{td}(G) \leq t\}.$$

Then $(\mathscr{T}_t^\star / \equiv, \operatorname{dist})$ is a compact metric space, in which $\mathscr{T}_t$ is dense.

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Recursive definition

The tree-depth can be computed inductively by:

$$\mathrm{td}(G) = \begin{cases} \max_H \ \mathrm{td}(H), & (H \text{ connected component of } G) \\ 1 + \min_v \ \mathrm{td}(G - v), & (G \text{ connected, } v \text{ vertex of } G) \\ 0, & \text{if } G \text{ is empty} \end{cases}$$

$\Longrightarrow$ can be considered as a game

- selection/deletion;
- cops/robber (Giannopoulou, Hunter and Thilikos, 2011).

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The selection/deletion game

- Alice selects a connected subgraph;
- Buddy deletes a vertex in the subgraph;
- Alice wins if G is not empty after k steps. Otherwise, Buddy wins.

Alice has a winning strategy

$k<\mathrm{td}(G)$

SD-game

$k\geq\mathrm{td}(G)$

Buddy has a winning strategy

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(rooted forest)

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SD-game $\xrightarrow{k\geq\mathrm{td}(G)}$ Buddy has a winning strategy (rooted forest)

Shelter

Definition (Giannopoulou, Hunter and Thilikos; 2011)

A shelter of a graph G is a collection $\mathscr{S}$ of non-empty subsets of vertices of G, ordered by $\subseteq$, such that $\forall A \in \mathscr{S}$:

- $G[A]$ is connected;
- either A is minimal, or

$$\forall x \in A \quad \exists B \in \mathscr{S} \text{ covered by } A \text{ such that } x \notin B.$$

$\rightarrow$ A rooted forest defines a strategy for Buddy;

A shelter defines a strategy for Alice.

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Paths and cycles

Lemma

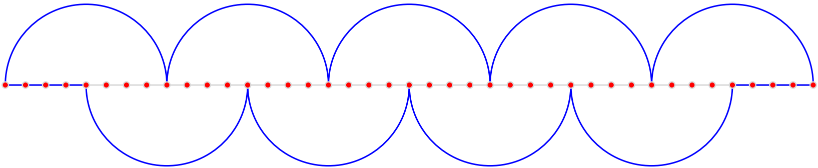
Let G be a connected graph, and let L be the length of a longest path of G. Then

$$\lceil \log_2(L+2) \rceil \leq \mathrm{td}(G) \leq L.$$

Lemma

Let G be a biconnected graph, and let L be the length of a longest cycle of G. Then

$$1 + \lceil \log_2 L \rceil \leq \mathrm{td}(G) \leq 1 + (L-2)^2.$$



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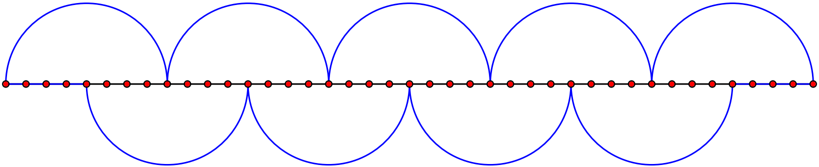

Algorithmic aspects

- No P approximation for $\mathrm{td}(G)$ with error $< |G|^{\varepsilon}$ (Bodlaender et al., 1995)
- Depth-First Search $\rightsquigarrow Y$ such that $G \subseteq \mathrm{Closure}(Y)$ and

$$\log_2(\mathrm{height}(Y) + 2) \leq \mathrm{td}(G) \leq \mathrm{height}(Y).$$

- Counting homomorphims from F to G in time

$$O(2^{|F|\,\mathrm{td}(G)}\,|F|\,\mathrm{td}(G)\,|G|).$$

- Homomorphism core in time $F(\mathrm{td}(G))\,|G|$
- Isomorphism in time $O(|G|^{\mathrm{td}(G)} \log |G|)$ (based on a standard vertex elimination order)

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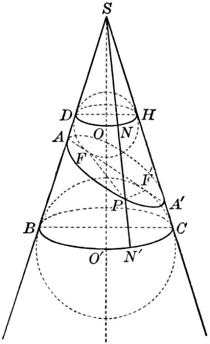
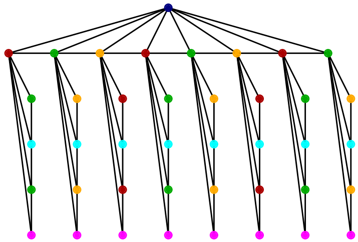
Sections



Principle

- Color the vertices of G by N colors,
- consider the subgraphs G_I induced by subsets I of $\leq p$ colors.



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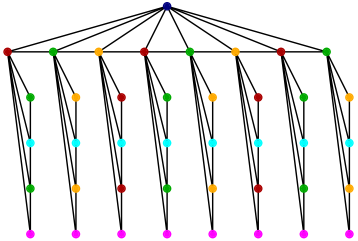

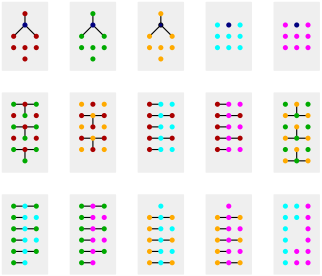

Low tree-width decompositions

Theorem (Devos, Oporowski, Sanders, Reed, Seymour, Vertigan; 2004)

For every proper minor closed class $\mathscr{C}$ and integer $p \geq 1$, there is an integer N, such that every graph $G \in \mathscr{C}$ has a vertex partition into N graphs such that any $j \leq p$ parts form a graph with tree-width at most $p-1$.

Remark

This theorem relies on Robertson-Seymour structure theorem.

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Low tree-depth decompositions

Chromatic numbers $\chi_p(G)$

$\chi_p(G)$ is the minimum of colors such that any subset I of $\leq p$ colors induce a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_p(G) \leq \cdots \leq \chi_{|G|}(G) = \mathrm{td}(G).$$

Countable graphs

A countable graph G has $\chi_p(G) \leq N$ if and only if $\chi_p(H) \leq N$ holds for every finite induced subgraph H of G.

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Let $\mathscr{C}$ be an infinite class of graphs.

Theorem (Nešetřil and POM, 2006)

$$\sup_{G\in\mathscr{C}} \chi_p(G) < \infty \qquad \iff \qquad \mathscr{C} \text{ has bounded expansion.}$$

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$$\forall p,\ \limsup_{G\in\mathscr{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \qquad \iff \qquad \mathscr{C} \text{ is nowhere dense.}$$

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Bounds on χ_p

Theorem (Nešetřil, POM)

Let G be a graph and let p be an integer. Then

$$\nabla_p(G) \le (2p+1)\binom{\chi_{2p+2}(G)}{2p+2}$$

$$\chi_p(G) \le P_r(\widetilde{\nabla}_{2^{p-2}+1/2}(G))$$

Theorem (Nešetřil, POM; 2011)

For every graph F of order p with at least one edge, and every $0 < \varepsilon < 1$, there exists $c > 0$ such that for every graph G it holds

$$(\#F \subseteq G) > |G|^{\alpha(F)+\varepsilon} \quad \Longrightarrow \quad \chi_p(G) > c\,|G|^{\varepsilon/p}.$$

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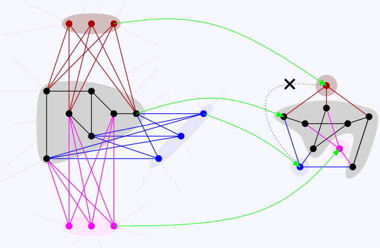
(k, F)-sunflowers

Definition

A (k, F)-sunflower $(C, \mathscr{F}_1, \dots, \mathscr{F}_k)$:



$$\forall X_1 \in \mathscr{F}_1, \dots \forall X_k \in \mathscr{F}_k$$

$$G[C \cup X_1 \cup \dots \cup X_k] \approx F$$

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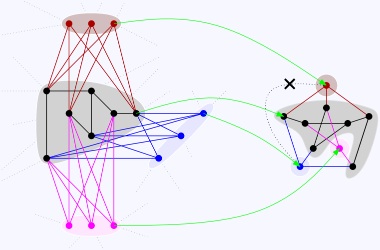

$$\forall X_1 \in \mathscr{F}_1, \dots \forall X_k \in \mathscr{F}_k$$

$$G[C \cup X_1 \cup \dots \cup X_k] \approx F$$

$$\Rightarrow k \leq \alpha(F) \text{ and } (\#F \subseteq G) \geq \prod_{i=1}^{k} |\mathscr{F}_i|.$$

Clearing & Stepping Up

Lemma (Nešetřil, POM; 2011)

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \varepsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\varepsilon}$ there exists in G a $(k+1, F)$-sunflower $(C, \mathscr{F}_1, \dots, \mathscr{F}_{k+1})$ with

$$\min_i |\mathscr{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\varepsilon}} \right)^{\tau(\varepsilon,p)}$$

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Proof.

- Consider a χ_p-coloring. Some section G_I contains $\binom{\chi_p(G)}{p}^{-1}$ proportion of the copies of F and has tree-depth $\leq p$;
- Encode F and G_I on colored forests of height p;
- Prove the lemma for colored forests by induction on the height.

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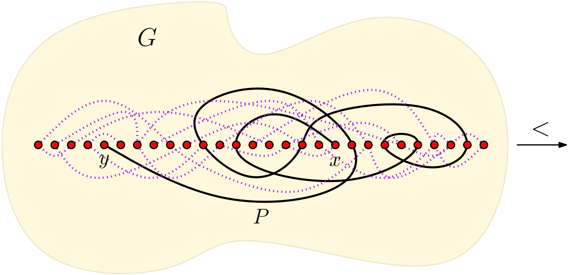
Hence $\exists G' \subseteq G$ such that

$$|G'| \geq (k+1) \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\varepsilon}} \right)^{\tau(\varepsilon,p)}$$

and $$(\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1} \right)^{k+1}.$$

Weak coloring



$$\mathrm{col}_k(G) \le \mathrm{wcol}_k(G) \le \mathrm{col}_k(G)^k \qquad \text{(Kierstead, 2003)}$$

$$\mathrm{wcol}_\infty(G) = \mathrm{td}(G) \qquad \text{(Nešetřil, POM)}$$

Weak coloring

Theorem (Zhu, 2008)

Let G be a graph, let $k \in \mathbb{N}$ and let $p = (k-1)/2$.

- $\nabla_p(G) + 1 \leq \mathrm{wcol}_k(G)$,
- *If $\nabla_p(G) \leq m$ then $\mathrm{col}_k(G) \leq 1 + q_k$, where q_k is defined as $q_1 = 2m$ and for $i \geq 1$, $q_{i+1} = q_1 q_i^{2i^2}$.*

Theorem (Zhu, 2008)

For every graph G, $\chi_p(G) \leq \mathrm{wcol}_{2^{p-1}}(G)$.

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Algorithmic version of LTDD theorem

Procedure A

for $k = 1$ **to** $2^{p-1} + 1$ **do**

Compute a fraternal augmentation.

end for

Compute depth p transitivity

Greedily color vertices according to the augmented graph

Theorem (Nešetřil, POM; 2008)

Procedure A computes a χ_p-coloring of G with $N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+\frac{1}{2}}(G))$ colors in time $O(N_p(G)\,|G|)$.

Remark

Also in time $O(2^p\,|G|^2)$.

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Problems

Checking first-order properties

Theorem (Nešetřil, POM)

Existential first-order properties may be checked in

- $O(n)$ *time for* G *in a class with bounded expansion,*
- $n^{1+o(1)}$ *time for* G *in a nowhere dense class.*

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First-order definable H-colorings

Definition

H-coloring is first-order definable in $\mathscr{C}$ if $\exists$ formula $\Phi(H)$ such that

$$\forall G \in \mathscr{C}: \qquad (G \to H) \quad \Longleftrightarrow \quad (G \models \Phi(H)).$$

Theorem (Nešťřil, POM; 2008)

If $\mathscr{C}$ has bounded expansion then for every connected F there exists H such that H-coloring is first-order definable on $\mathscr{C}$ and equivalent to non-existence of a homomorphism from F.

Problem

Let $\mathscr{C}$ be hereditary, addable, closed by subdivisions.
Assume that $\forall g \in \mathbb{N}$, $\exists H$ non bipartite with odd-girth $> g$ such that H-coloring is first-order definable in $\mathscr{C}$. Is it true that $\mathscr{C}$ has bounded expansion?

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Graphs ε-close from being very simple

Hyperfinite graphs

Assume $\mathscr{C}$ has bounded Δ and sublinear separators and let $\varepsilon > 0$.
$\exists N\ \forall G \in \mathscr{C}\ \exists F \subset E(G)$: $|F| < \varepsilon|G|$ and $G - F$ has no connected component of order $> N$.

Corollary of Devos, Oporowski, Sanders, Reed, Seymour, Vertigan; 2004

Assume $\mathscr{C}$ excludes some minor and let $\varepsilon > 0$.
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Appendix

Infinite trees

Definition (Tree)

- A tree is a poset $(T, <)$ such that for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$.
- For each $t \in T$, the order type of $\{s \in T : s < t\}$ is the height of t.
- The height of T is the least ordinal greater than the height of each element of T.
- T is rooted (single-rooted) if it contains a single t (the root of T) with height 0.

tree-depth of infinite graphs

Assuming the axiom of choice, $\mathrm{td}(G)$ exists and

$$|V(G)| = \aleph_\alpha \implies \mathrm{td}(G) \leq \omega_\alpha.$$

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