9^{TH} TUTORIAL ON RANDOMIZED ALGORITHMS

Coupling of Markov chains

1. *Random walk on a hypercube.* In the hypercube graph of dimension d, vertices are binary strings of length d and two vertices are connected by an edge iff they differ in exactly one coordinate. For $d = 2$, the graph is

$$\Big(\{00, 01, 10, 11\}, \Big\{\{00, 01\}, \{00, 10\}, \{11, 01\}, \{11, 10\}\Big\}\Big) .$$

We start at 0^d and do the following random walk:

- With probability $1/2$ we stay at the current vertex.
- With probability $1/2$ we choose an index $j \in [d]$ uniformly at random and flip the j-th bit.

The Markov chain is ergodic (finite, aperiodic, and irreducible), so it converges to a unique stationary distribution (which one is it?). Show that the random walk has

$$\tau(\varepsilon) \le d \ln(d/\varepsilon) .$$

2. *Sampling colorings.* Let $G = (V, E)$ be a graph with maximum degree Δ. We would like to uniformly sample a proper coloring with c colors. We use the following Markov chain: Given a proper coloring, pick $v \in V$ and a color $\ell \le c$ uniformly at random and color v using ℓ if this new coloring is proper.

a) Show that this Markov chain is ergodic for $c \ge \Delta + 2$. What is the stationary distribution? What breaks down for $c \le \Delta + 1$?

b) Analyze the mixing time for $c \ge 4\Delta + 1$ using a simple coupling (pick the same v and ℓ in both chains). Hint: look at random variable d_t = the number of vertices with different colors in the two chains.

c) Finally, we will show a better coupling that works even for $c \ge 2\Delta + 1$ (thus saving a factor of two).