7^{TH} TUTORIAL ON RANDOMIZED ALGORITHMS

Approximate counting

1. Consider the following variant of the *Coverage algorithm* for approximating the DNF counting problem. For $t = 1, \ldots, N$,

- select a clause C_t at random with probability proportional to the number of satisfying truth assignments,
- select a satisfying truth assignment a for C_t uniformly at random, and
- define random variable $X_t = 1/|\text{cov}(a)|$, where $\text{cov}(a)$ denotes the set of clauses thatare satisfied by a (there's always at least one).

Our estimator for $\#F$ (the number of satisfying assignments for the DNF formula) is

$$Y = \sum_{t=1}^{N} \frac{\sigma}{N} \cdot X_t \,,$$

where σ is the sum of the sizes of the coverage sets $\text{cov}(a)$ over all satisfying assignments a. Prove that Y is an (ε, δ)-approximation for $\#F$ for a sufficiently large N.

2. *Parity of perfect matchings.* Show an algorithm that given a bipartite graph G (partites consisting of the same number of vertices) determines if the number of perfect matchings is even or odd.

3. *Fraction of approximations.* We say that $\hat{x}$ is an ε-approximation of x iff

$$(1 - \varepsilon)x \leq \hat{x} \leq (1 + \varepsilon)x$$

Show that for $\varepsilon < 1/2$, if we have ε-approximation $\hat{s}$ of a number s and ε-approximation $\hat{t}$ of a number t, then $\hat{s}/\hat{t}$ is an 4ε-approximation of s/t. (It's sufficient to determine the upper bound.)

4. *Product of approximations.* Let $\varepsilon > 0$ be fixed. Find a suitable choice of $\overline{\varepsilon}$ such that if we take $\overline{\varepsilon}$-approximations $(\hat{a}_i)_{i=1}^{n}$ of numbers $(a_i)_{i=1}^{n}$, then $\prod_{i=1}^{n} \hat{a}_i$ is an ε-approximation of $\prod_{i=1}^{n} a_i$. (It's sufficient to work out the upper bound.)

5. *Main course: Counting matchings.* Let $G = (U \cup V, E)$ be a bipartite graph where $|U| = |V| = n$ and $\delta(G) > n/2$. We define:

$m_k =$ the number of matchings of size k in G, and
$r_k = m_k/m_{k-1} =$ the fraction of the # of k-matchings to the # of $k-1$-matchings.

Let $\alpha \geq 1$ be a real number such that $1/\alpha \leq r_k \leq \alpha$. Pick $N = n^7\alpha$ elements from $M_k \cup M_{k-1}$ independently uniformly at random (approximately uniform generation covered in the lecture). Set $\hat{r_k}$ to the fraction of observed k-matchings to $(k-1)$-matchings. Show that

$$\left(1 - 1/n^3\right) r_k \leq \hat{r_k} \leq \left(1 + 1/n^3\right) r_k$$

with probability at least $1 - \exp(-n)$. (Hint: use the Estimator theorem from the lecture.)

Then show why accurate approximations of r_k's are useful for estimating the number of perfect matchings.