6^{TH} TUTORIAL ON RANDOMIZED ALGORITHMS

1. Let $G = (V, E)$ be a d-regular graph (so its largest eigenvalue is $\lambda_1 = d$). Assume that the absolute value of any other eigenvalue is at most λ for $0 \leq \lambda \leq \lambda_1$. Then for every $S \subseteq V$ with $|S| = \alpha \cdot n$, it holds that

$$|e(S, \overline{S}) - d \cdot (1 - \alpha) \cdot \alpha \cdot n| \leq \lambda \cdot \alpha \cdot (1 - \alpha) \cdot n\,,$$

where $e(S, \overline{S})$ is the number of edges between S and $\overline{S} = V \setminus S$.

2. Let μ be a probability distribution, that is $\|\mu\|_1 = 1$ and $\mu_j \geq 0$ (for each $j \in \Omega$). Let us define $d(\mu, \nu)$ the distance of two probability distributions as:

$$d(\mu, \nu) = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|$$

Show that:

$$d(\mu, \nu) = \max_{A \subseteq \Omega} \mu(A) - \nu(A)$$

where $\mu(A) = \sum_{x \in A} \mu(x)$. (This is called the *total variation distance.*)

3. Show that for any $v \in \mathbb{R}^n$ it holds that

$$\frac{1}{\sqrt{n}} \|v\|_1 \leq \|v\|_2 \leq \|v\|_1$$

(Hint: for the first inequality, use scalar product of v with a suitable vector to get $\|v\|_1$)

4. Consider the following variant of the Coverage algorithm for approximating the DNF counting problem. For $t = 1, \ldots, N$,

- select a clause C_t at random with probability proportional to the number of satisfying truth assignments,
- select a satisfying truth assignment a for C_t uniformly at random, and
- define random variable $X_t = 1/|\text{cov}(a)|$, where $\text{cov}(a)$ denotes the set of clauses that are satisfied by a (there's always at least one).

Our estimator for $\#F$ (the number of satisfying assignments for the DNF formula) is

$$Y = \sum_{t=1}^{N} \frac{\sigma}{N} \cdot X_t\,,$$

where σ is the sum of the sizes of the coverage sets $\text{cov}(a)$ over all satisfying assignments a. Prove that Y is an (ε, δ)-approximation for $\#F$ for a sufficiently large N.