NDMI025 – Randomized Algorithms
(Pravděpodobnostní algoritmy)
LS 2015 – Jiří Sgall

Homework 2 – March 12

Due date: March 30 before the lecture

Recitation session for this homework set will take place on April 2

Sign all the sheets either by your name or by your nickname.

(1) A lollipop is an undirected graph on n vertices consisting of a complete graph on t vertices and a path on $s = n - t$ additional vertices, connected by an edge from one endpoint to one of the vertices of the complete graph. Let u be some other vertex of the complete graph (i.e., some vertex of degree $t-1$) and v be the other endpoint of the path (i.e., the single vertex of degree 1). Consider the random walk on this graph. Compute the hitting time h_{uv}, i.e., the expected time of reaching v from u, as well as h_{vu}.

Determine also the values of t for which the hitting times are maximized or minimized.

(2) A cat and a mouse each independently take a random walk on a cycle of length $2n+1$. (In each step, they both take a random step, i.e., both random walks are synchronized.) The game ends (the cat eats the mouse) if they land at the same vertex at the same time. (If they traverse the same edge in the opposite directions, nothing happens.) Find the expected number of steps of the game if the cat and the mouse start (a) at adjacent nodes and (b) at distance n (i.e., at the largest possible distance).

(3) Let T be a complete ternary tree of depth d with majority gates at the inner nodes and $n = 3^d$ distinct variables at the leaves. The tree defines a Boolean function $T(\vec{x})$ given by successive evaluation of the subtrees. (Think of it as of a Boolean formula with a ternary majority connective.)

(a) Consider a recursive randomized algorithm $A(u)$ that evaluates the value of the vertex u as follows:

If u is a leaf, query the corresponding variable and return this value.

If u is an inner node, choose randomly (independently of other choices) two children v and w of u and evaluate recursively $A(v)$ and $A(w)$. If the results are equal, output this value. Otherwise evaluate $A(z)$ for the third child and output this value.

Find the expected running time of $A(r)$ for the root r (and the worst-case input). I.e., derive an upper bound on the expected running time and also find an input on which this bound is tight.

(b) Prove that every deterministic algorithm that correctly evaluates $T(\vec{x})$ on all inputs has to query the value of all the variables at the leaves.

(4) Let G be a connected d-regular undirected graph on n vertices and let λ_i be the ith largest eigenvalue of the adjacency matrix of G. (Count the eigenvalues with multiplicities.) Prove that the following three statements are equivalent:

(a) G is bipartite,

(b) $\lambda_n = -\lambda_1$, and

(c) for every $i = 1, \ldots, n$, we have $\lambda_{n-i+1} = -\lambda_i$.

Bonus question: Is the same true for graphs that are not regular?