Graph theory

Note: Strange structure due to finals topics. Will improve later.

Graph coloring

T:A d-degenerate graph can be coloured with $d+1$ colours.

P:Greedy.

T(Brooks):A graph can be coloured using Δ colours if it's not an odd cycle or a complete graph.

P:The choosability version of this follows from ERT theorem.

T(Vizing):A graph is edge-colorable using either Δ or $\Delta+1$ colours.

P:Induction on $||G||$. For a coloring of $G-uv$, one colour is missing in $\delta(u)$ and one in $\delta(v)$. A maximal graph on those two color classes has a path from u to v, otherwise done.

Choose vertices xy_0. Find a maximal set of vertices of $\delta(x)$ such that the colour missing at y_0 is the colour of the edge xy_1, the colour missing at y_1 is present on the edge xy_2 etc.

Take last xy_k and its missing colour β. Create Kempe chain to some y_i to x. Switch to y_i, find Kempe chain again. Contradiction.

T(Thomassen):Every planar graph is 5-choosable.

P:Every face bounded by a triangle, outer face a cycle. Suppose one vertex is coloured 1, other 2 and the rest on the outer face have $L(v)=3$ and the inside vertices $L(v)=5$. Then G colourable.

Apply induction, look for chord, remove vertices if outer face chordless.

Erdos-Rubin-Taylor

D:G is a *Gallai block* $\equiv$ G is K_n or C_{2k+1}.

D:G is a Gallai tree $\equiv$ connected and (maximal) Gallai blocks form a tree.

O:Graph connected, lists of degree-size, one bigger. Then G is colorable.

R:1-connected graph with articulation can be coloured.

T(Erdos-Rubin-Taylor):G connected. G degree-choosable $\leftrightarrow$ $G\neq$ Gallai tree.

L(ERT lemma):G2−conn., G not a Gallai tree block. Then $\exists v\in G$ and $v'\ /v''\in\delta(G)$ s.t. $G-v'-v''$ conn.

P(ERT lemma): G 3-connected: pick any non-adjacent vertices.

G only 2-conn.: Pick a x,y cut such that the minimal component is minimal.

- $xy\in||G||$: Pick v,v' two neighbours of y. Use Menger's theorem to assert victory.
- $xy\notin||G||$: Choose the same v',v'' neighbors of y. Define Z_x: vertices connected to x after removing v',v''. Define Z_y analogously.
  Consider $C_0\equiv$ the smallest $G-x-y$. Both Z_x and Z_y have empty intersection with it, or we're done.
  In the end, only v' is connected to x and y, apply induction on $H=G/v'$. Case analysis based on whether or not is H complete.

L(Application of ERT Lemma):G 2-connected, not a Gallai block. Then it is deg-choosable.

P:Assume there are different lists $L(u),L(v)$. We can find u,v that are neighbors with this property. Think of u as root. Colour the rest of the graph greedily because v suddenly has more colours than neighbors.

So there are no such lists. All the vertices are of the same degree. If it's an odd cycle, we are done. If not, use ERT lemma and find two vertices and use the greedy argument.

P(Erdos-Rubin-Taylor): G connected. Proving stronger assertion: If G is not connected, it is a Gallai tree and each block has a "designated" list which it all shares.

By induction on block size. One block: okay. More blocks: cut off corner block, proceed by induction.

R(Brooks):If a graph G is connected, it can either be coloured with lists of size *deg* or it is a K_l or a C_{2k+1}.

L(Kernel lemma): Let G be a graph with associated lists for colouring. If G has an orientation with $d^+(v)<|S_v|$ everywhere, and this orientation also has a kernel for every induced subgraph, then H can be coloured by S_v.

P: Induction. Take a colour α, find all vertices that have it in their lists, find their kernel. Colour kernel, remaining vertices still fulfill conditions, induce.

T(Galvin):$\chi'(G)=\chi'_L(G)$ for bipartite graphs.

P:We need to find an orientation for the line graph sasisfying the lemma. Take two partitions. Take k-coloring. Assign orientation based on colouring, and based on where the two edges meet (X or Y). Prove condititons.

Q(Hadwiger):k-colorability implies a minor of K_k.

Proven for 5, 6.

Q(List Coloring Conjecture):Edge choosability and edge colorability coincide.

Perfect graphs

D:A *perfect graph* is a graph which has $\forall G\subseteq_{ind}G':\chi(G')=\omega(G')$.

O:Bipartite graphs, interval graphs, chordal graphs are perfect.

T(Weak Perfect Graph Theorem):G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect.

T(Strong Perfect Graph Theorem):G is perfect $\Leftrightarrow$ contains no odd hole or antihole (of size ≥ 5).

Regular graphs

T(Moore):G d-regular without K_3,K_4 and with exactly $n=d^2+1$ vertices: $d\in 2,3,7,17$.

O:For a Moore graph and its adjacency matrix, $A^2=J-A+(d-1)I$.

P(Moore):We know that $Sp(J)=\{n,0^{d-1}\}$. Because of observation, the eigenvalues also obey the polynomial property. Therefore, we have $\lambda^2++\lambda-(d-1)1=n$ or 0.

For the main eigenvalue d, we get the condition $d^2+1=n$. For the remaining conditions the right side is zero. Solving it as a quadratic equation, we get solutions for λ_1 and λ_2 which are not d. The discriminant will be $\sqrt{4d-3}$. Condition on it being rational or not, and you get the remaining numbers.

T(Turan):$\forall r>1,n$ every graph without a K^r subgraph and $ex(n,K^r)$ edges is a Turan graph T^{r-1}.

P:Among complete k-partite graphs, Turan graphs have the most edges. Among Turan graphs, T^{r-1} has the most edges. We need to prove that a graph with $ex(n,K^r)$ edges is a complete multipartite graph.

If not, non-adjacency is not equivalence, and so find three conflicting vertices. Duplicating some yields the contradiction.

O:$t_{r-1}(n)\leq n^2/2\frac{r-2}{r-1}$.

D:Density $d(A,B)=||(A,B)||/|A||B|$.

D:An *ε-regular pair* (A,B) for given ε has the property that every set (X,Y) of size at least $\varepsilon X,\varepsilon Y$ respectively has density $d(X,Y)$ ε-close to $d(A,B)$.

D:An *ε-regular partition* for given ε satisfies the following:

- $|V_0|\leq\varepsilon|V|$.
- $|V_i|=|V_j|$.
- All but most εk^2 pairs are not ε-regular.

L(Regularity lemma):For every ε and every $m\exists M$ such that every graph of size $\geq m$ admits an ε-regular partition.

T(Erdos-Stone):For all $r\geq 2,s\geq 1,\varepsilon>0\exists n_0$ such that all graphs with more vertices and at least $t_{r-1}(n)+\varepsilon n^2$ edges contain K_r^s as a subgraph.

L(Removal lemma): TBD.

Graph connectivity

T(Menger):G graph. Then minimum number of vertices separating A from B in G is equal to the maximum number of disjoint $A-B$ paths in G.

P:Induction on $||G||$. If G has no k disjoint $A-B$ paths, contract one edge $e=xy$. G/e contains an $A-B$ separator Y of less than k vertices. One of them must be the contracted v_e. Therefore $Y-v_e+x+y$ must be a separator in G of exactly k vertices.

Consider now $G-e$. Every $A-X$ separator in $G-e$ is an $A-B$ separator in G, and so each such separator has at least k vertices. Therefore, there are k disjoint $A-X$ paths and k disjoint $B-X$ paths. These paths do not meet outside X, and can be extended to $A-B$ paths.

T(Mader):

L:There is a function h such that every graph of average degree $h(r)$ contains K^r as a topological minor.

P:We show by induction on $m=r\ldots\binom{r}{2}$ that every graph with average degree $\geq 2^m$ has a topological minor X with r vertices and m edges.

Induction start can be done easily with a maximal cycle. Now, consider $d(G)\geq 2^m$. Assume G connected. Find maximal set U s.t. $d(G/U)\geq 2^m$. Vertex with minimum degree is acceptable, so there exists such a nonempty set, and $\delta(U)\neq\emptyset$.

Take neighborhood $\delta(U)$. If there is any vertex with degree ¡ 2^{m-1}, we can add it. Thus, minimum/average degree is $\geq 2^{m-1}$. Apply induction. Take topological minor and connect two vertices through U, which is connected.

T:k-linked implies $f(k)$-connected.

P:Prove it for $f(k) = h(3k)+2k$. Find topological minor of K^{3k}. Use Menger on its vertices and $2k$ vertices S and T. Choose k vertices in K^{3k} that do not have endvertices on the $2k$ disjoint paths by Menger. Connect the paths.

Special properties of oriented graphs

Algebraic properties of graphs

Spectral theory

D:λ is an *eigenvalue* $\equiv \exists x : Ax = \lambda x$.

O:$Au - \lambda u = 0 \leftrightarrow det|A - \lambda I| = 0$.

O:A has a base of eigenvectors $\leftrightarrow \exists X : X^*X = E$ and $X^*AX = I * (\lambda_1, \lambda_2, \dots)$.

R:G graph, A its adjacency matrix $\rightarrow$ there are n different real eigenvalues of A.

R:G graph d-regular $\rightarrow \Lambda_{max} = d$.

O:There exist cospectral graphs, for example $K_{1,4}$ and $C_4 + v$.

Flows and Tutte polynomial

T(Recipe Theorem): Let $f(G, x, y)$ be a function on graphs such that for some α, β, γ, it satisfies the following recurences:

- $f(\overline{K_n}) = \gamma^n$,
- $f(G) = xf(G/e)$ if e is a loop,
- $f(G) = yf(G \setminus e)$ if e is a coloop (bridge),
- $f(G) = \alpha f(G/e) + \beta f(G \setminus e)$ otherwise.

Then it can be computed using the formula

$$f(G, x, y) = \gamma^{c(G)}\alpha^{r(G)}\beta^{r^*(G)}T(G, x/\alpha, y/\beta).$$

O:Chromatic polynomial can be expressed as $P(G; z) = z^{c(G)}(-1)^{r(G)}T(G, 1 - z, 0)$.

O:Flow polynomial can be expressed as $F(G; k) = (-1)^{r^*(G)}T(0, k - 1)$.

Matching theory

T(Hall):Bipartite graph has PM iff $\forall A \subseteq X : \delta(A) \geq |A|$. **P:**Use Menger or mincut-maxflow on G. If there is not a perfect matching, there exist $A \subseteq X$ and $B \subseteq Y$ such that $|A| + |B| < |X|$ there is no edge from $X \setminus A$ to $Y \setminus B$. Then $\delta(X \setminus A) \subseteq B$ and $|\delta(X \setminus A)| \leq |B| < |X| - |A| = |X \setminus A|$.

T(Tutte):A graph has a 1-factor iff $\forall S : q(G \setminus S) \leq |S|$, where q counts the number of odd components.

P:Choose S_0 maximal such that the inequality becomes an equality. Prove then that even components have a PM, odd components without a vertex have a PM and finally using Hall that S_0 to $\{C|C \text{ odd }\}$ has a PM.

Edmonds' Algorithm: Find an Edmonds forest, contract flowers (odd cycles).

Ramsey theory

D:$[X]^k$ is a set of all k-tuples of X.

T(Infinite Ramsey):Let k, c be positive numbers, X infinite. If we color $[X]^k$ by c colors, we find a monochromatic infinite subset.

P:Prove it by induction on k, with c fixed. Construct a series of sets X_i and associated elements x_i such that:

- $X_{i+1} = X_i - x_i$.
- All k-sets $x_i \cup Z$ with Z from X_{i+1} are monochromatic with colour that we associate with x_i.

Just pick x_i arbitrarily and note that if you associate the colour of x_i to $k-1$-sets of $X_i - x_i$, you can apply induction on k and get condition 2.

Since x_i is an infinite set, at least one colour occurs infinitely many times, and this colour is the monochromatic subset.

T(Finite Ramsey): For every k, c, r there exists $n \geq k$ such that all $[n]^k$ colourings with c colours contain a monochromatic subset of size r.

P:If not, for all $n \geq k$ there is a bad colouring. Use Konig's Infinity Lemma (compactness) and extend the bad colourings into an infinite one. This contradicts infinite Ramsey.

T(Erdos-Szekeres):For $k \geq 2, l \geq 2$, every non-degenerate set of $\binom{k+l-4}{k-2} + 1$ points in the plane contains a k-cup or a l-cap. Also for less points, a set avoiding this exists.

P:Induction. Suppose that there is a set of $\binom{k+l-4}{k-2} + 1$ points an it has neither. Choose L the list of last points of $k-1$-cups. Then $X \setminus L$ has no $k-1$ cup or l cap. Apply induction. L is therefore big enough and (by induction) contains a $l-1$ cap.

T(Van-der-Waarden):Given k, r there exists n_0 such that if we colour all sequences of length $\geq n_0$ with k colours, we get a monochromatic arithmetic sequence of length r.

T(Hales-Jewett):$\forall a$ (cube size) and r (color number) there exists $N = HJ(a, r)$ such that colouring the cube A^N with r colours produces a monochromatic combinatorial line.

P(VdW from HJ):Use $f((x_1, x_2, \dots, x_n)) = \sum_i x_i$. Given colouring of $[ft]$, colour element $\in K$ with the colour $c(f(x_1, x_2 \dots))$. Note that any combinatorial line is an arithmetic sequence.

Infinite combinatorics

T:Every connected graph contains a spanning tree.

P:Apply Zorn's lemma on a set of finite inclusion-wise ordered spanning trees.

T(König's Infinity Lemma):Let $V_0, V_1, \dots,$ be an infinite sequence of non-empty disjoint finite sets of vertices. G graph of their union. Assume that every vertex in V_i has a neighbor in V_{i-1}. Then G contains a ray that traverses all the sets.

P:

Infinite Ramsey.

Structural properties of set systems

T(Sperner's Theorem): Any set system on n elements has all antichains of size $\leq \binom{n}{n/2}$.

P:Follows from Bollobas's theorem below.

T(Dilworth):

T(Sunflower lemma):If a s-regular set system $F \geq s!(k-1)^s$, then it contains a sunflower with k petals.

P:Induction. Base case simple. If there is a subset of k mutually disjoint sets, we win, so assume there is not. Apply induction and win.

T(Erdos-Ko-Rado): Ground set $n \geq 2k$. If a k-regular set system F has an intersection, then $|F| \leq \binom{n-1}{k-1}$.

P:In PM cheatsheet.

D: Define $n(k, l)$ as maximal n s.t. $\exists A_{[1,n]}$ & $B_{[1,n]}$ s.t.

1. $|A_i| = k, |B_i| = l$,
2. $A_i \cap B_i = \emptyset$,
3. $i \neq j \rightarrow A_i \cap B_j \neq \emptyset$.

T(Bollobás' lemma on intersections): $n(k, l) = \binom{k+l}{k}$.

P:In PM cheatsheet.

T(Kruskal-Katona):