1./

Knots and links

KNOT : subset of $\mathbb{R}^3$ homeomorphic to circle S^1

LINK : several disjoint knots

AMBIENT ISOTOPY taking knot K to knot K' : continuous map $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that h_0 is identity map and $h_1(K) = K'$

TAME KNOTS : ambient isotopic to simpled closed polygons (piecewise linear simple closed curves)

REGULAR PROJECTION of knot onto plane:

- finitely many multiple points
- these points are double points (marking CROSSINGS)

DIAGRAM of a knot or link : regular projection together with underpass/overpass indicated at each double point —

left overpasses right

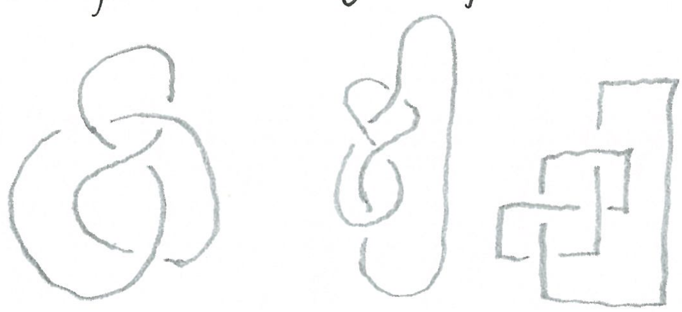
left underpasses right

left trefoil

right trefoil

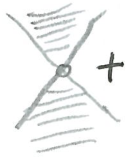
Hopf link
(2 components)

Borromean rings
(3 components)

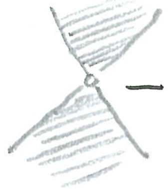


← three diagrams of Figure of Eight knot

diagram D ⟷ plane 4-regular $\tilde{D}$
face 2-coloured, vertex signs


⟷


left over right (from point of view of black)


⟷


right over left

<u>Example:</u>

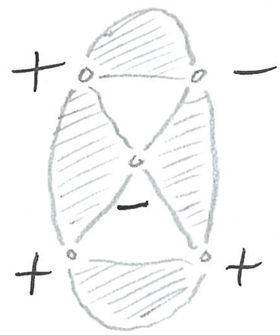

⟷


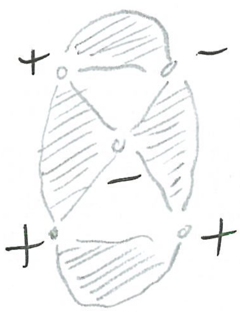
diagram D
with face 2-colouring
(outer face white)

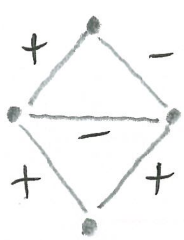
face 2-coloured vertex-signed
plane 4-regular $\tilde{D}$

---

face 2-coloured vertex-signed plane 4-regular $\tilde{D}$ ⟷ edge-signed plane graph G

medial ←



⟷


3.

Lemma Bijection between edge-signed plane graphs G and link diagrams D

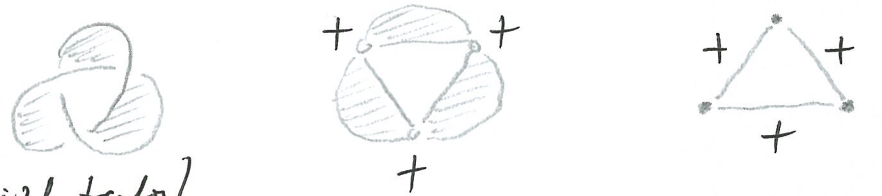
(G is the TAIT GRAPH of link L with diagram D)

ALTERNATING LINK : crossings alternate under - over

— L is alternating iff in its Tait graph G all signs are same

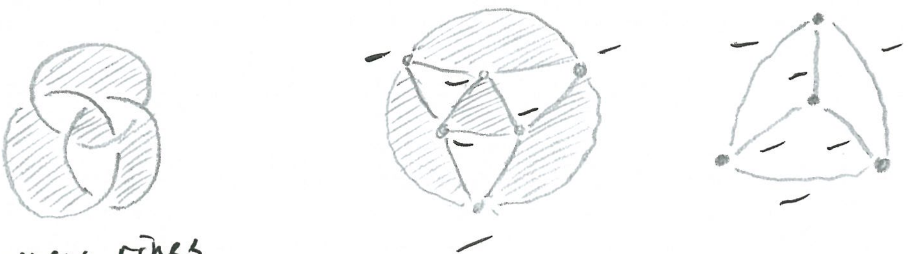
examples

left trefoil D $\tilde{D}$ Tait graph G



right trefoil

Hopf link

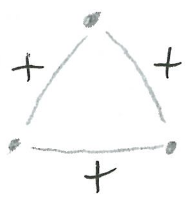


Borromean rings

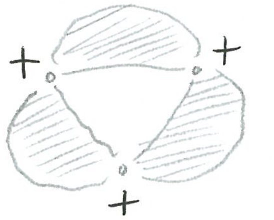
In fact every link with crossing number ≤ 7 is equivalent to an alternating link (with minimum number of crossings)

Prop^n $\mathcal{G}$ Tait graph of link L,
$\mathcal{G}^*$ plane dual of $\mathcal{G}$ with signs reversed
⟶ $\mathcal{G}^*$ is Tait graph of a link equivalent to L

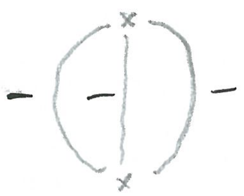
example:



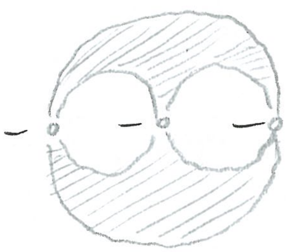
Tait graph $\mathcal{G}$



right trefoil



dual $\mathcal{G}^*$ with opposite signs



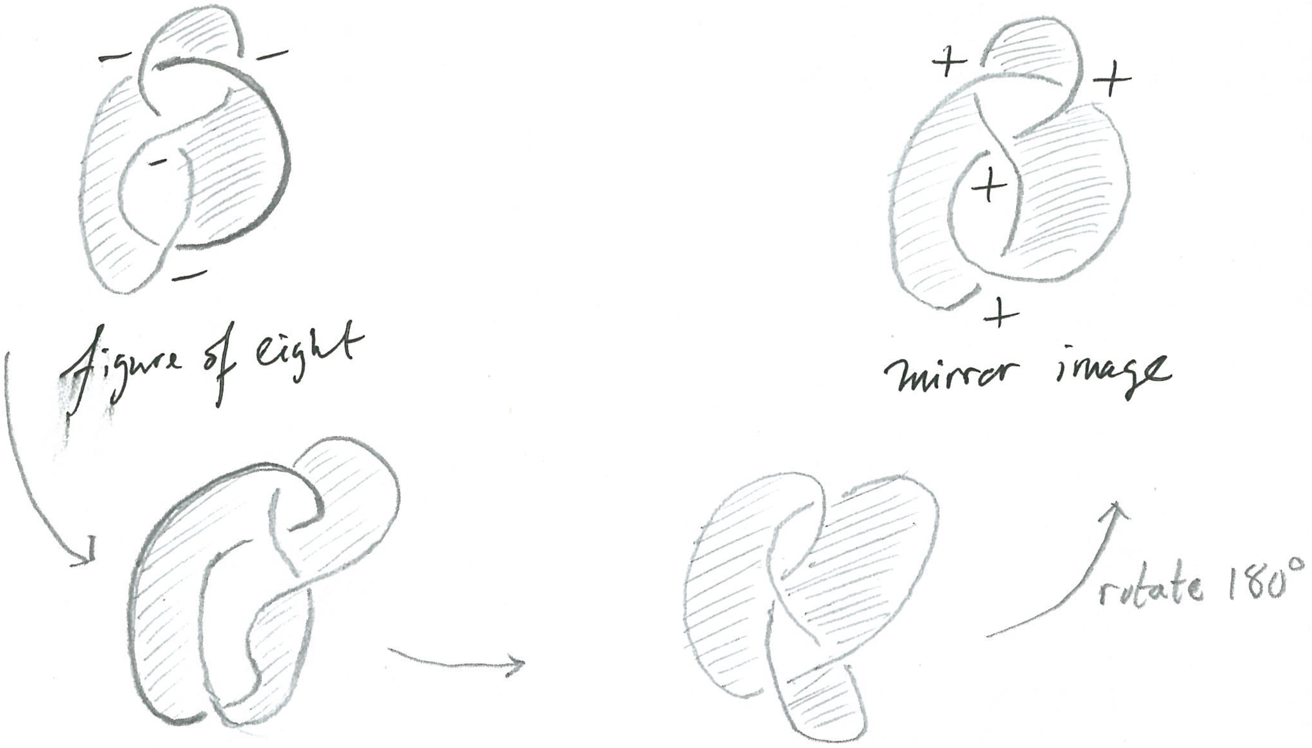
MIRROR IMAGE of a link: interchange overcrossings with undercrossings, i.e., switch signs in Tait graph

example: left trefoil is mirror image of right trefoil.

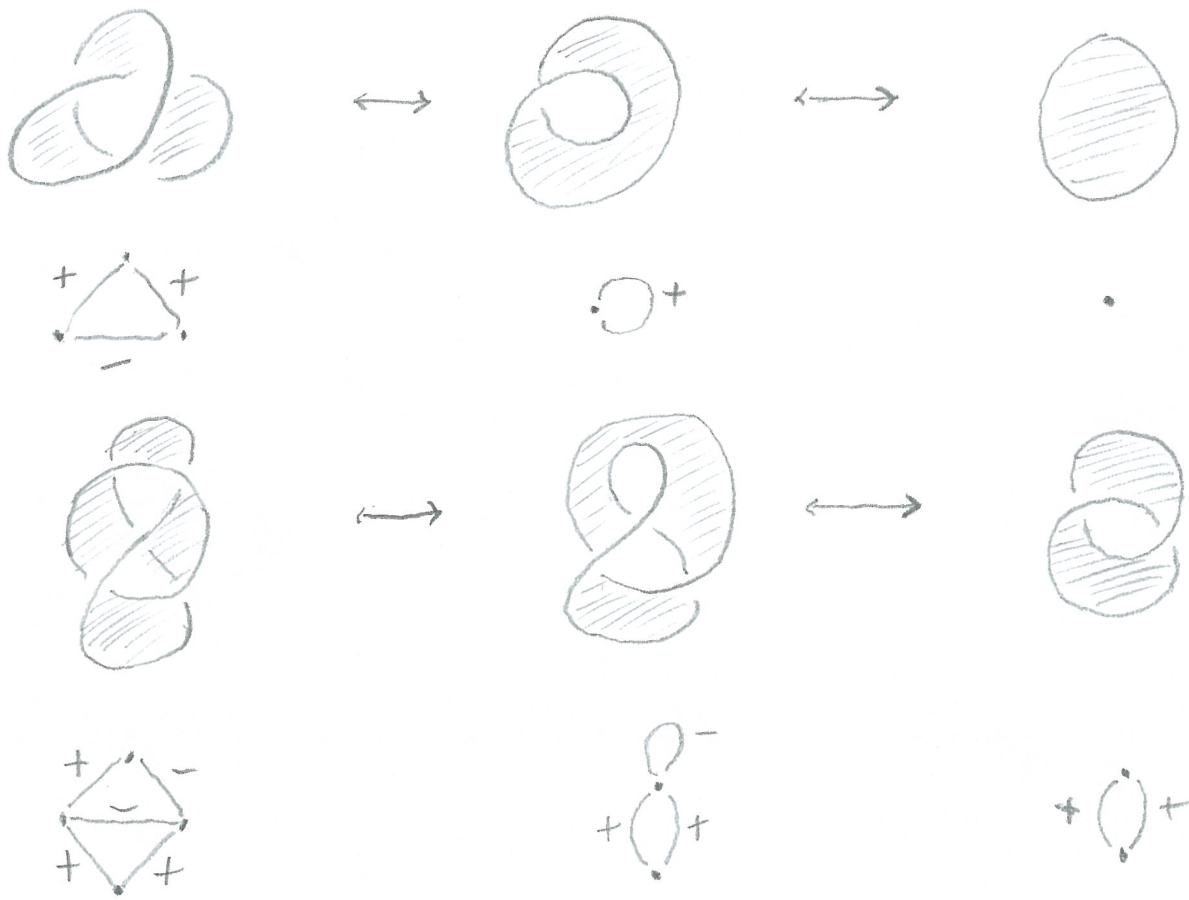
Def^n link is AMPHICHEIRAL (or ACHIRAL) if ambient isotopic to its mirror image.

5./

example: Figure of Eight is amphicheiral:

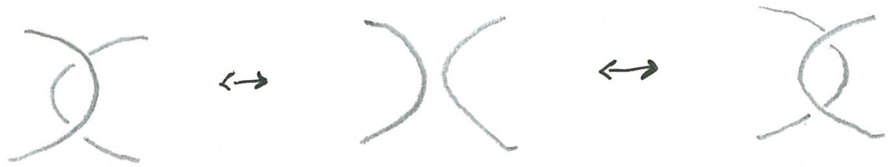


Equivalence of Links, Reidemeister Moves



REIDEMEISTER MOVES on a link diagram:

(I) 

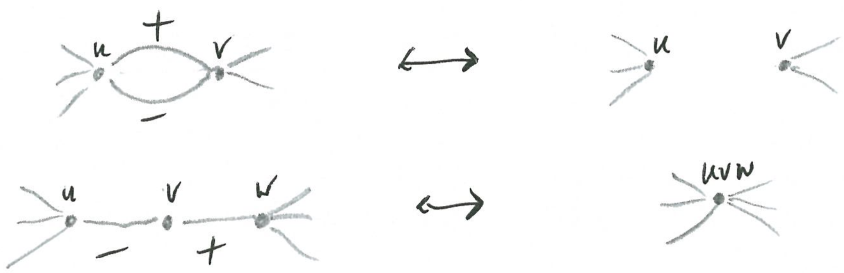
(II) 

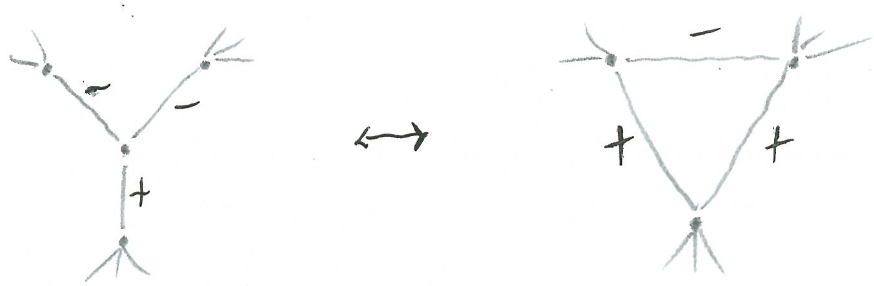
(III) 

Theorem (Reidemeister, 1935)
Two links are ambient isotopic if and only if the diagram of one can be transformed to the other by a finite sequence of Reidemeister moves (I), (II), (III).

Reidemeister moves on Tait graph:

(I) 

(II) 

(III) 

(star-triangle operation)

Problem Complexity status of deciding whether a given knot is equivalent (ambient isotopic) to the unknot?
(size of input: number of crossings, ie number of edges in Tait graph)

Definition LINK INVARIANT f has property that $f(L) = f(L')$ whenever L and L' are ambient isotopic.

examples:
- number of component knots of a link
- minimum number of crossings in a diagram representing a link

Knowing $f(L) \neq f(L')$ for link invariant f means that L and L' are not ambient isotopic. In particular, $f(L) \neq f(\text{unknot})$ tells us that L is not equivalent to the unknot, and $f(L) \neq f(\bar{L})$ where $\bar{L}$ is the mirror image of L tells us that L is chiral (not amphicheiral).

By Reidemeister's Theorem, f is a link invariant if and only if it is invariant under Reidemeister moves (I), (II) and (III).

The Kauffman bracket

defined on links in terms of their diagrams:

(K1) $\langle \bigcirc \rangle = 1$ $\bigcirc$ = unknot

(K2) $\langle \bigcirc \cup D \rangle = d \langle D \rangle$ disjoint union

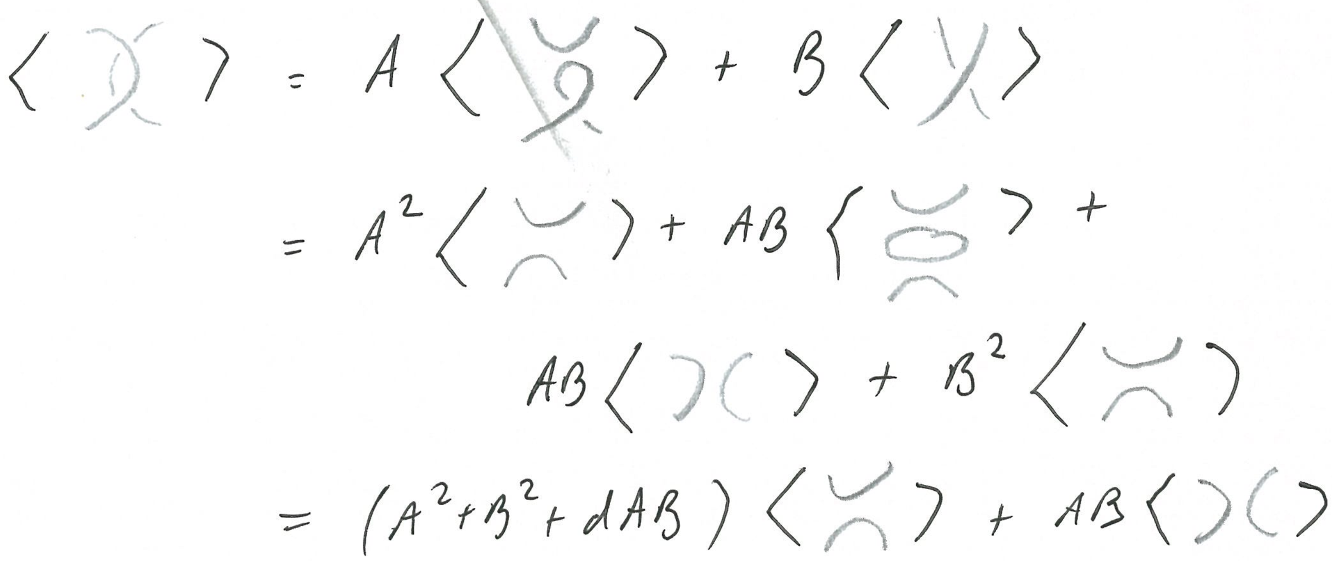
(K3) 

A, B, d commuting indeterminates

(K3) reduces the number of crossings by one.

Example: 

Invariance under Reidemeister move (II): 

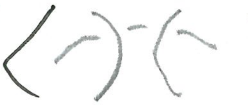


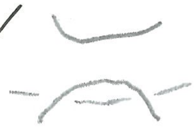
in order that  need $AB = 1$, $A^2+B^2+dAB = 0$

i.e. $B = A^{-1}$ $d = -(A^2+A^{-2})$

9./ Invariance under Reidemeister move (III):  ... 

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$\underset{\text{by (II)}}{=} A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

and $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$\underset{\text{by (II)}}{=} A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

Hence $\langle$  $\rangle = \langle$  $\rangle$

Reidemeister move (I): 

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= -A^{-3} \langle$  $\rangle$

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= -A^{3} \langle$  $\rangle$

So not invariant under (I). To fix this move to ORIENTED LINKS.

Side-note:  $\langle$  $\rangle = -A^{-3} \langle O \rangle = -A^{-3}$

 $\langle$  $\rangle = -A^{3}$

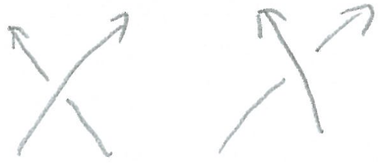
10.

$\langle$  $\rangle = -A^{3} \langle \bigcirc \rangle = -A^{3}$

$\langle$  $\rangle = -A^{-3} \langle \bigcirc \rangle = -A^{-3}$

ORIENTED LINK : for each component knot give a direction of traversal

assign weight to crossings :



right trefoil

right trefoil will reverse orientation

for a knot, weight is independent of orientation — a reversal of orientation reverses local orientation on both segments.

for a link, different components can be independently reoriented so there is a change of sign weight.

Def^n For diagram D of oriented link, writhe $(D) = w(D) =$ sum of sign weights on crossings

Self-writhe $(D) = w(D_1) + \cdots + w(D_k)$

when D is link with components $D_1, \ldots, D_k$

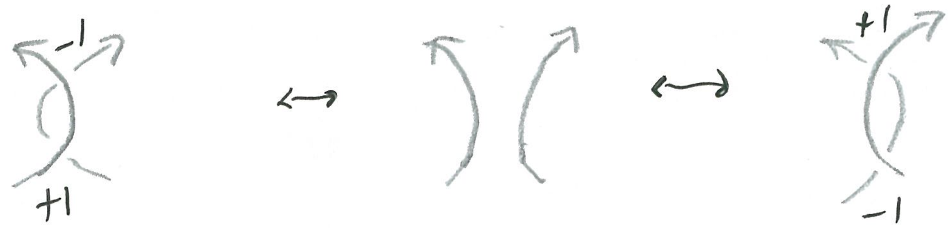
$= s(D)$

note : $s(D)$ is independent of orientation of D : so can be defined for unoriented D

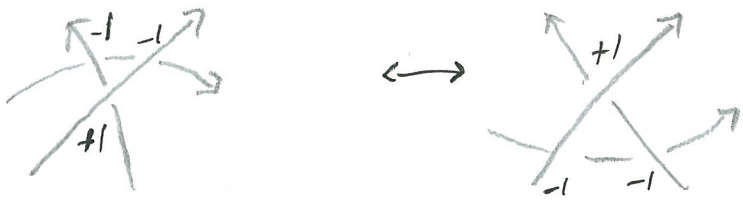
11.

Lemma $w(D)$ and $s(D)$ are regular isotopy invariants (invariant under Reidemeister moves (II) and (III))

Pf: (II)



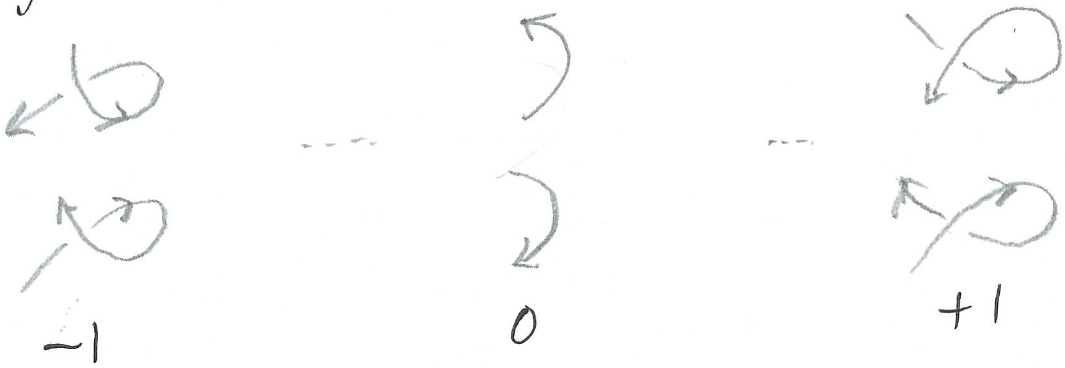
(III)



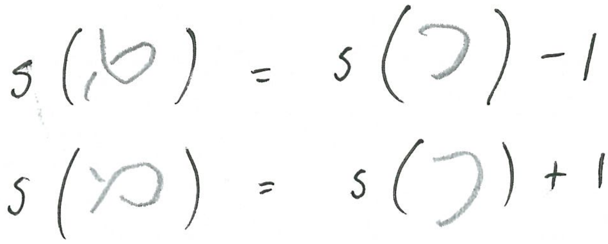
and similarly for other orientations

Theorem The Laurent polynomial $f[D] = (-A)^{-3s(D)} \langle D \rangle$ in $\mathbb{Z}[A, A^{-1}]$ is an invariant of ambient isotopy for unoriented links. (KAUFFMAN POLYNOMIAL)

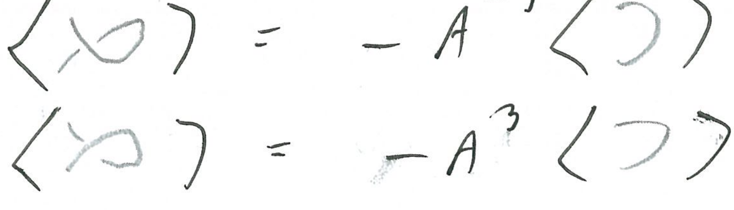
Pf: Since $\langle D \rangle$ and $s(D)$ are invariant under (II) and (III) only need check invariance under (I).

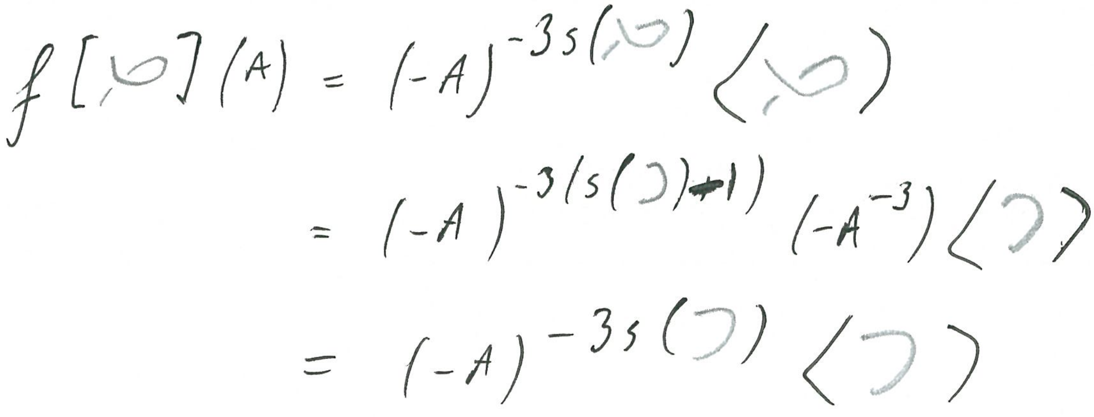


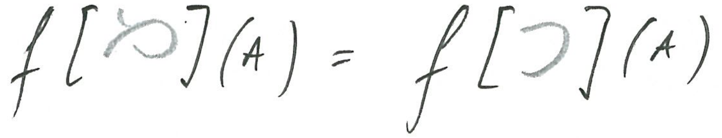
so:



and we have (from previous)



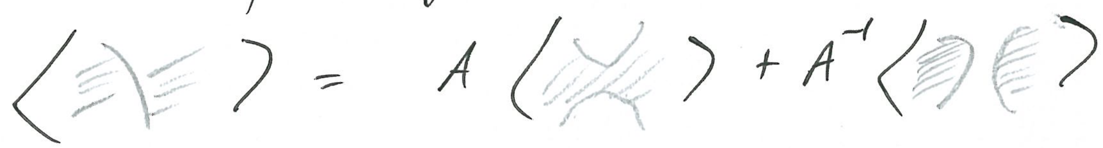
Hence 

Similarly,  □

Prop^n if $\bar{L}$ is mirror image of L then

$$f[\bar{L}](A) = f[L](A^{-1})$$

Pf: From recursive definition of bracket

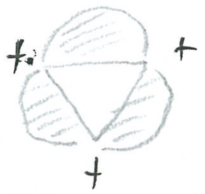


mirror: 

Also writhe weights are switched in mirror image □

Kauffman polynomial for alternating links

L alternating link



$\tilde{D}$ diagram
all crossings same sign

Tait graph G
all edges same sign

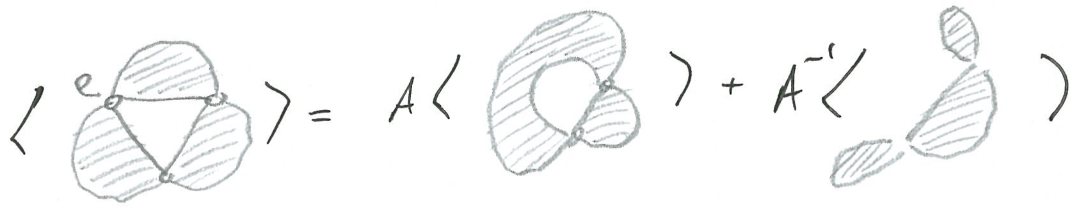
Take all signs + in plane graph G
to give Tait graph of alternating link

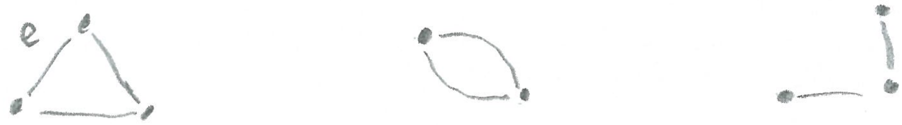
Recursion (K3)

on diagram D: 

on (Tait) graph G : $\langle G \rangle = A \langle G/e \rangle + A^{-1} \langle G \backslash e \rangle$
↳ plane

<u>Example</u>:





Also,  $= -A^{-3}$  $= -A^{3}$

So by Recipe Theorem for Tutte polynomial:

Theorem For (alternating link) diagram D with Tait graph the plane graph $G = (V, E)$ with all edges +

$$\langle D \rangle = A^{2|V| - |E| - 2}\, T(G; -A^{-4}, -A^{4})$$

$G = (V, E, F)$ faces

$$= A^{|V| - |F|}\, T(G; -A^{-4}, -A^{4})$$

where $T(G)$ is the Tutte polynomial of plane graph G, and mirror image $\bar{D}$ has

$$\langle \bar{D} \rangle = A^{|V^*| - |F^*|}\, T(G^*; -A^{-4}, -A^{4})$$

where $\mathcal{G}^* = (V^*, E^*, F^*)$ is plane dual of $\mathcal{G}$

Corollary Kauffman polynomial of alternating link with diagram D is given by

$$f[D](A) = (-A)^{-3s(D)} \langle D \rangle$$

$$= (-1)^{|E|} A^{|V|-|F|-3s(D)} T(\mathcal{G}; -A^{-4}, -A^{4})$$

Pf: E = vertices of D = edges of $\mathcal{G}$

Let E_+ be those weight $+1$ in arbitrary orientation of link with diagram D

E_- " -1

$|E| = |E_+| + |E_-|$ $\quad$ $s(D) = |E_+| - |E_-|$

so $(-1)^{s(D)} = (-1)^{|E|}$ $\quad\square$

__Example__ right trefoil has diagram with Tait graph K_3 (all edges $+$), writhe $= 3$

$$T(K_3; x, y) = x^2 + x + y$$

$\langle$  $\rangle = A^{2.3-3-2}(A^{-8} - A^{-4} - A^{4})$

$= A^{-7} - A^{-3} - A^{5}$

and $f[$  $](A) = -A^{-16} + A^{-12} + A^{-4}$

Mirror image is left trefoil (with $\mathcal{G}$ =  $= K_3^*$)

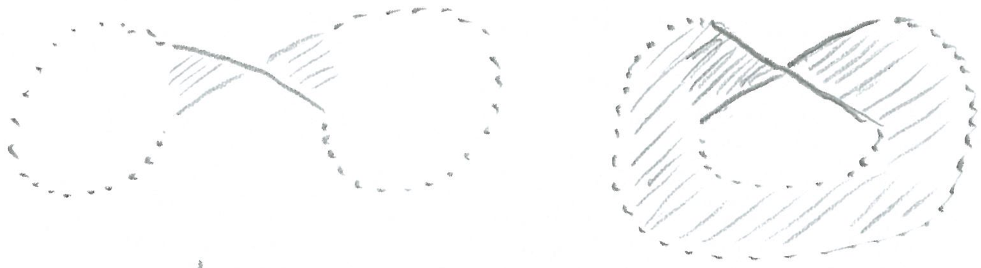
and $f[$  $](A) = -A^{16} + A^{12} + A^{4} \neq f[$  $](A)$

implying left and right trefoils inequivalent, i.e. __chiral__

15. Tait conjecture

Def^n A crossing is NUGATORY if some two of the local regions appearing at the crossing are parts of the same region in the whole diagram.

A nugatory crossing in an alternating link appears as a bridge or loop in the Tait graph.



Theorem (conj. Tait, proved Murasugi & Thistlethwaite ca. century later)
The number of crossings of a connected alternating link diagram without nugatory crossings is an ambient isotopy invariant.

Pf: diagram D with Tait graph all signs + (take mirror image if all signs −), plane graph G this without signs.

Laurent polynomial $f[D](A) = (-A)^{-3s(D)} \langle D \rangle$
has span maximum degree − minimum degree := span(D)

$\langle D \rangle = A^{2|V|-|E|-2}\, T(G; -A^{-4}, -A^{4})$ has

$\text{span} \langle D \rangle = 4(r(G) + |E| - r(G)) = 4|E|$
$= 4 \#$ crossings of D (edges G ↔ vertices medial ↔ crossings of D)

when G has no loops or bridges (i.e. no nugatory crossings in D)

Lemma The Tutte polynomial $T(G;x,y)$ of a bridgeless, loopless graph $G=(V,E)$ has degree $r(G)$ in x and degree $|E|-r(G)$ in y. *

Pf. (By induction on $|E|$.) When $E=\emptyset$, $T(G;x,y)=1$ and $r(G)=0=|E|$.
The recurrence $T(G)=T(G/e)+T(G\backslash e)$ for $e\in E$ (not a bridge or loop by hypothesis) and

$$r(G/e)=\begin{cases} r(G)-1 & \\ r(G) & e \text{ loop} \end{cases}$$

: $|E(G/e)|-r(G/e) = |E|-r(G)$ for non-loop e

$$r(G\backslash e)=\begin{cases} r(G) & \\ r(G)-1 & e \text{ bridge} \end{cases}$$

provides the inductive step. □

---

* moreover $T(G;x,y) = x^{r(G)} + y^{|E|-r(G)}$ + lower degree terms

so that span of $T(G;-A^{-4},-A^{4})$ is equal to $4(|E|-r(G)+r(G)) = 4|E|$.