Structural Graph Theory
DocCourse 2014: Lecture Notes

Andrew Goodall, Lluís Vena (eds.)

Structural Graph Theory DocCourse 2014
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Preface

The DocCourse “Structural Graph Theory” took place in the autumn semester of 2014 under the auspices of the Computer Science Institute (IÚUK) and the Department of Applied Mathematics (KAM) of the Faculty of Mathematics and Physics (MFF) at Charles University, supported by CORES ERC-CZ LL-1201 and by DIMATIA Prague. The schedule was organized by Prof. Jaroslav Nešetřil and Dr Andrew Goodall, with the assistance of Dr Lluís Vena. A web page was maintained by Andrew Goodall, which provided links to lectures slides and further references.[1]

The Structural Graph Theory DocCourse followed the tradition established by those of 2004, 2005 and 2006 in Combinatorics, Geometry, and Computation, organized by Jaroslav Nešetřil and the late Jiři Matoušek, and has itself been followed by a DocCourse in Ramsey Theory in autumn of last year, organized by Jaroslav Nešetřil and Jan Hubička.

For the Structural Graph Theory DocCourse in 2014, five distinguished visiting speakers each gave a short series of lectures at the faculty building at Malostranské námestí 25 in Malá Strana: Prof. Matt DeVos of Simon Fraser University, Vancouver; Prof. Johann Makowsky of Technion - Israel Institute of Technology, Haifa; Dr. Gábor Kun of ELTE, Budapest; Prof. Michael Pinsker of Technische Universität Wien/ Université Diderot - Paris 7; and Dr Lenka Zdeborová of CEA & CNRS, Saclay. The audience included graduate students and postdocs in Mathematics or in Computer Science in Prague and a handful of students from other universities in the Czech Republic and abroad.

Parallel with the special lecture series, Dr Andrew Goodall lectured on “Counting flows on graphs: finite Abelian groups and integer flows”, as part of the regular IÚUK/ KAM course “Vybrané Kapitoly z Kombinatoriky I” (Selected Chapters in Combinatorics). Students taking the course were encouraged as an alternative to end-of-semester exams to write a project based on material from those DocCourse lectures that particularly interested them.

The lecture notes that follow were kindly provided by the speakers subsequent to the course, with some light editing by Andrew Goodall and Lluís Vena, who set this booklet in its present form.

Jaroslav Nešetřil, Andrew Goodall and Lluís Vena
Prague, April 2017

[1] http://iuuk.mff.cuni.cz/~andrew/DocCourse2014.html

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DIMATIA

CORES

Vybrané Kapitoly z Kombinatoriky I/ Selected Chapters in Combinatorics NDMI055
Guarantor: J. Nešetřil, A. Goodall
Assistant: L. Vena Cros

Doc-Course, October - December 2014

Charles University in Prague

Thurs 9 Oct (S6, 12:20 & 14:30) & Mon 13 Oct (S6, 14:00)
Prof. M. DeVos
Simon Fraser University, Vancouver

Fri 10 Oct (S6, 14:00) & Thurs 16 Oct (S6, 12:20 & 14:10)
Prof. J. Makowsky
Technion - Israel Institute of Technology, Haifa

Mon 20 Oct (S6, 14:00) & Thurs 23 Oct (S6, 14:00)
Dr G. Kun
ELTE, Budapest

Thurs 23 Oct (S6, 12:20) & Mon 3 Nov (S6, 14:00) & Thurs 6 Nov (S6, 12:20 & 14:10)
Prof. M. Pinsker
Technische Universität Wien/ Université Diderot - Paris 7

Mon 10 Nov (S6, 14:00) & Tues 11 Nov (S7, 14:00) & Thurs 20 Nov (S6, 12:20)
Dr L. Zdeborová
CEA & CNRS, Saclay

http://iuuk.mff.cuni.cz/events/doccourse2014/
Computer Science Department (IÚUK) & Department of Applied Mathematics (KAM),
Faculty of Mathematics and Physics (MFF), Malostranské nám. 25, Prague 1, Czech Republic

Titles & Abstracts

Prof. Matt DeVos

Immersion for 2-regular digraphs

In this talk we will focus on the world of 2-regular digraphs, i.e. digraphs for which every vertex has indegree and outdegree equal to 2. Surprisingly, this family of digraphs behaves under the operation of immersion in a manner very similar to the way in which standard graphs behave under minors. This deep truth is best evidenced by the work of Thor Johnson, who developed an analogue of the Robertson-Seymour Graph Minor Theory for 2-regular digraphs under immersion. We will discuss some recent work together with Archdeacon, Hannie, and Mohar in this vein. Namely, we establish the excluded immersions for certain surface embeddings of 2-regular digraphs in the projective plane.

Flows in bidirected graphs

Tutte showed that for planar dual graphs G and G^*, a k-coloring of G is equivalent to the existence of a nowhere-zero k-flow in G^*. This led him to his famous conjecture that every bridgeless graph has a nowhere-zero 5-flow. Although this conjecture remains open, Seymour has proved that every such graph has a nowhere-zero 6-flow. Bouchet studied this flow-coloring duality on more general surfaces, and this prompted him to introduce the notion of nowhere-zero flows in bidirected graphs. He conjectured that every bidirected graph without a certain obvious obstruction has a nowhere-zero 6-flow. Improving on a sequence of earlier theorems, we show that every such graph has a nowhere-zero 12-flow.

Average degree in graph powers

For a graph G and a positive integer k, we let G^k denote the graph with vertex set $V(G)$ and two vertices adjacent in G^k if they have distance at most k in the original graph G. Motivated by some problems in additive number theory (which we will explain), we turn our attention to determining lower bounds on the average degree of the graph G^k when the original graph G is d-regular. We will describe fairly complete answers to this question when $k < 6$ and in general when k is congruent to 2 (mod 3). This talk represents joint work with McDonald, Scheide, and Thomassé.

Prof. Johann Makowsky

Classical graph properties and graph parameters and their definability in SOL

Intriguing graph polynomials. Why is the chromatic polynomial a polynomial? Comparing graph polynomials. Connection matrices and their use in showing non-definability.

Dr Gábor Kun

Expanders everywhere

I will give the most important equivalent definitions of expanders. I plan to highlight many different applications from group theory to graph theory, computer science and number theory. I would like to mention some basic ideas of the proof of the Banach-Ruziewicz problem, the Jerrum-Sinclair algorithm and, if time allows, the Bourgain-Gamburd-Sarnak sieve.

Prof. Michael Pinsker

Algebraic and model-theoretic methods in constraint satisfaction

The Constraint Satisfaction Problem (CSP) of a finite or countable first-order structure S in a finite relational language is the problem of deciding whether a given conjunction of atomic formulas in that language is satisfiable in S. Many classical computational problems can be modelled this way. The study of the complexity of CSPs involves an interesting combination of techniques from universal algebra, Ramsey theory, and model theory. This tutorial will present an overview over these techniques as well as some wild conjectures.

Dr Lenka Zdeborová

Coloring random and planted graphs: Thresholds, structure of solutions, algorithmic hardness

Random graph coloring is a key problem for understanding average algorithmic complexity. Planted random graph coloring is a typical example of an inference problem where the planted configuration corresponds to an unknown signal and the graph edges to observations about the signal. Remarkably in a recent decade or two tremendous progress has been made on the problem using (principled, but mostly non-rigorous) methods of statistical physics. We will describe the methods - message passing algorithms and the cavity method. We will discuss their results - structure of the space of solutions, associated algorithmic implications, and corresponding phase transitions. We will conclude by summarizing recent mathematical progress in making these results rigorous and discuss interesting open problems.

Immersion and embedding of 2-regular digraphs
Flows in bidirected graphs
Average degree of graph powers

Matt DeVos

Simon Fraser University, Burnaby, BC Canada

Editors' note: The lecture notes that follow are on three topics, the second and third of which the reader may explore further in the references given:

1. Immersion and embedding of 2-regular digraphs.
2. Flows in bidirected graphs [1]
3. Average degree of graph powers [2]

1 Immersion and embedding of 2-regular digraphs

1.1 Introduction

In this section we will be interested in 2-regular digraphs (i.e. digraphs for which every vertex has both indegree and outdegree equal to 2). There is a natural operation called splitting which takes a 2-regular digraph and reduces it to a new 2-regular digraph. To *split* a vertex v with inward edges uv and $u'v$ and outward edges vw and vw', we delete the vertex v and then add either the edges uw and $u'w'$, or we add the edges uw' and $u'w$. If G, H are 2-regular digraphs, we say that H is *immersed* in G if a graph isomorphic to H may be obtained from G by a sequence of splits.

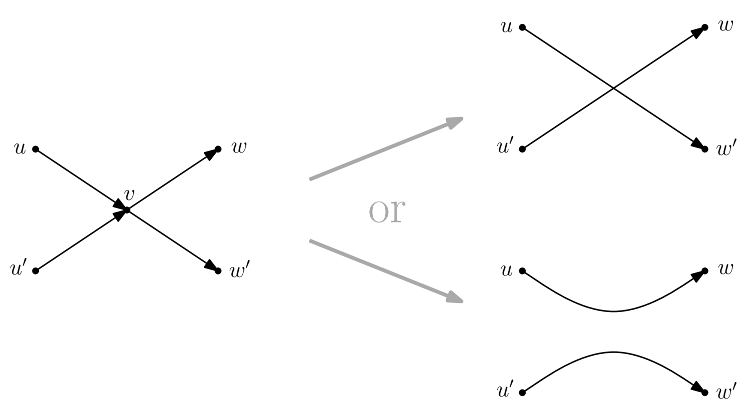

Figure 1: splitting a vertex

Our central goal in this article will be to show how the theory of 2-regular digraphs under immersion behaves similar to the theory of (undirected) graphs under graph minor operations. We will begin with some motivation. Consider an ordinary undirected graph G which is embedded in an orientable surface. The *medial* graph H is constructed from G by the following procedure. For every edge e of G add a vertex of the graph H in the centre of e. Now, whenever two edges $e, f \in E(G)$ are consecutive at a vertex (or equivalently, consecutive along a face) we add an edge between the corresponding vertices of H. Based on this construction, every vertex of the original graph is contained in the centre of a face of the medial graph, and every face of the original graph completely contains a new face of the medial graph. So, the faces of the medial graph have a natural bipartition into these two types, and indeed this gives a proper 2-face colouring of our medial graph. Since our surface is orientable, we may direct the edges so that every face containing a vertex of the original graph is oriented clockwise. The following figure gives an example of this oriented medial graph.

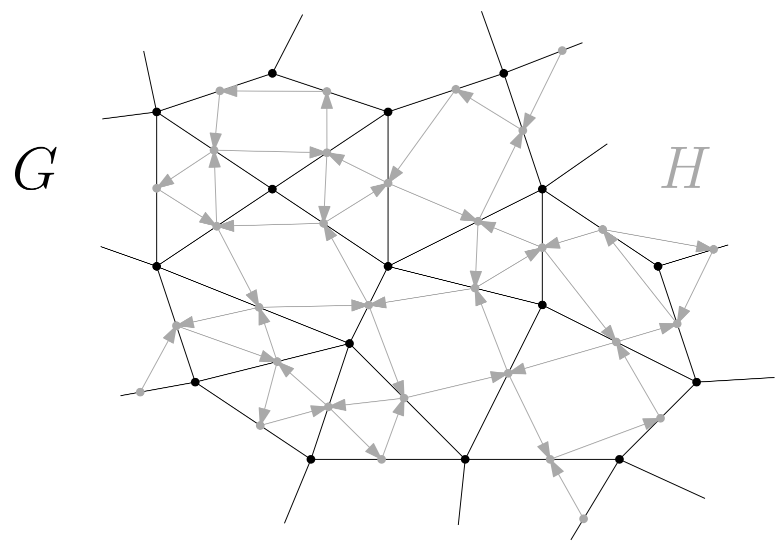

Figure 2: A graph G and its oriented medial H

Let us first note that this oriented medial graph is a 2-regular digraph. Now let's consider how the medial graph H changes when we delete an edge e of the original graph. Suppose that v is the vertex of the medial graph which corresponds to e. After deleting e the new medial graph will no longer have the vertex v, and (check this!) in fact, the new oriented medial may be obtained from the original by splitting v. Similarly, if we modify the original graph by contracting e, the new medial may be obtained by doing the other split at v. So, in this setting, we see that our minor operations on G correspond precisely to splitting vertices of the oriented medial. This connection suggests a general study of 2-regular digraphs under immersion, and this will be our direction going forward.

There is a key feature of the embedded 2-regular digraphs coming from the aforementioned construction. Namely, at each vertex v in this embedding, the cyclic order of the incident edges goes inward-outward-inward-outward.

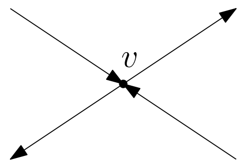

Figure 3: a nice local rotation

As you can easily see, if this is the local behaviour at v, then either of the possible ways of splitting v will result in a new 2-regular digraph which still has a natural embedding in the surface. Motivated by this, let us now define a *special* embedding of a 2-regular digraph to be one which satisfies this property at every vertex. Now, similar to the behaviour of (undirected) graphs under minor operations, we have the following easy observation for our 2-regular digraphs.

Observation 1.1. *If G is a 2-regular digraph embedded in a surface $\mathcal{S}$, then every digraph immersed in G also embeds in $\mathcal{S}$.*

The Graph Minors project of Robertson and Seymour established a number of very deep properties of (undirected) graphs under the relation of minors. One great achievement of this project is a rough structure theorem for the class of graphs not containing a fixed graph H as a minor. A remarkable consequence of this is that every proper minor closed class of graphs (ex. planar graphs) is characterized by a finite list of excluded minors (i.e minor minimal graphs not in the class). A Ph.D. student of Seymour named Johnson proved an analogous rough structure theorem for 2-regular digraphs under immersion (which strongly features special embeddings). He claims to know a proof of the finite list of excluded immersed graphs, but this was never written.

One very pleasing property of 2-regular digraphs is that their behaviour under immersion is somewhat cleaner and simpler than that of usual graphs under minors. As evidence for this, we offer the following chart which gives information about the number of minor minimal graphs not embeddable in certain surfaces, and the analogous list of immersion-minimal graphs with no special embedding. This theorem for the plane will be given in the next section and I'm unclear who deserves credit for it (probably either Johnson or Seymour). The projective plane theorem for 2-regular digraphs is due to Archdeacon, D., Hannie, and Mohar. The same group expects to have the corresponding result for the torus shortly, and I have optimistically filled this entry.

Surface	Minors (graphs)	Immersion (2-reg. digraphs)
plane	$K_{3,3}$, K_5 (Kuratowski, Wagner)	$C_3^{(2)}$
proj. plane	35 graphs (Archdeacon)	$C_3^{(2)} + C_3^{(2)}$, $C_3^{(2)} \cdot C_3^{(2)}$, $C_4^{(2)}$, C_6^2
torus	$> 10^4$ graphs, unsolved	hopefully coming soon!

To explain our notation here, let us assume G and G' are digraphs. Then $G^{(2)}$ is the digraph obtained from G by adding a new edge in parallel with each existing edge, and G^2 is the digraph obtained by adding a new edge from u to v whenever there is a directed path of length 2 from u to v. The disjoint union of G and H is denoted $G+G'$ and we let $G \cdot G'$ denote a digraph obtained from the disjoint union of G and G' by choosing edges $(u, v) \in E(G)$ and $(u', v') \in E(G')$, deleting them and then adding the edges (u, v') and (u', v). Finally, we let C_k denote a directed cycle of length k.

1.2 Planar Embeddings

Our goal in this section is to prove the following result.

Theorem 1.2. *A 2-regular digraph has a special embedding in the plane if and only if it does not immerse $C_3^{(2)}$.*

Proof. First we prove the "only if" direction. By Observation 1.1, it suffices to show that $C_3^{(2)}$ has no special embedding in the plane. To see this, first note that in a special embedding, every face is bounded by a closed directed walk. Since these directed walks must use every edge exactly twice, every special embedding of $C_3^{(2)}$ has at most 4 faces. So, by Euler's formula, if we have a special embedding of $C_3^{(2)}$, the Euler characteristic of the surface must be at most $3 - 6 + 4 = 1$.

For the "if" direction, we may assume that our digraph G is connected. Choose an Euler tour W of G, let $v \in V(G)$ and consider the behaviour of the tour W at v. The tour W must pass through v twice, say using the edges (u, v) then (v, w) and later using the edges (u', v) then (v, w'). Now modify the graph G to by uncontracting a new (undirected) edge at the vertex v forming the two adjacent vertices v, v' so that we now have the directed edges (u, v), (v, w) and (u', v'), (v', w').

If we do this at every such vertex, we obtain a mixed graph (with both directed and undirected edges) which we call H. The graph H has a directed cycle containing every vertex as given by our original Euler tour. We shall view H as drawn with this cycle as a circle and all other edges as chords. So, we will think of each vertex of the original graph as a chord of this circle.

Based on this figure, we now construct a new graph K with vertex set $V(G)$ and an edge between u, v if the chords corresponding to u and v cross. This type of graph is known as a *circle graph*. We now split into cases depending on whether K is bipartite. If K is a bipartite graph, then we may partition our chords into two sets $\{A, B\}$ so that no two in the same set cross. Based on this we can embed the graph H in the plane by putting the chords in A on the inside and the chords in B outside of the circle. Once we have an embedding of H, we can contract all of these chords to obtain a special embedding of our original graph G.

The remaining possibility is that K is not bipartite, and in this case we may choose an induced odd cycle $C \subseteq K$. For every vertex v of the original graph which is not in $V(C)$ split the vertex v in accordance with the Euler tour (i.e. so if the edges (u, v) and (v, w) appear consecutively in the tour, we split v to add the edge (u, w)). Let G' be the 2-regular digraph obtained by doing this for every vertex not in $V(C)$, and let W' be the Euler tour obtained from W. Using the same process as before, we let H' be the mixed graph obtained from G' by uncontracting, and let K' be the corresponding circle graph. Observe that by this operation, the resulting graph K' is precisely C.

If our cycle $C = K'$ has length > 3 then we will modify it to make it shorter by two. To do this, we simply choose two consecutive vertices on our cycle and split them in the original graph G' in a manner not in accordance with our Euler tour W'. The reader may verify that the resulting 2-regular digraph, say G'' will have an associated circle graph K'' which is still a cycle but is now two vertices shorter. By repeating this process, we may obtain a 2-regular graph G^* immersed in G with the property that the circle graph K^* associated with G^* is a triangle. It follows that G^* is the digraph $C_3^{(2)}$, as desired. $\square$

1.3 Peripheral Cycles

Although our result for the projective plane isn't terribly complicated, it does require some preliminary lemmas, most of which are quite sensible and meaningful. In this section we will sketch a proof of one of these tools.

For an undirected graph G, we say that a cycle C is *peripheral* if there is no edge $e \in E(G) \setminus E(C)$ with both ends in $V(C)$, and the graph $G - V(C)$ is connected. If G is embedded in the plane, then it is easy to see that every peripheral cycle must be the boundary of a face.

Theorem 1.3 (Tutte)**.** *If G is a 3-connected graph, then every edge is in at least two peripheral cycles.*

Corollary 1.4. *A 3-connected planar graph has a unique embedding in the plane.*

We will prove an analogous theorem for 2-regular digraphs. In such a digraph G, we define a directed cycle C to be *peripheral* if $G - E(C)$ is strongly connected. If G is any 2-regular graph which has a special embedding in the plane, then deleting the edges of any directed cycle separates the part of the graph inside this cycle from the outside. So, as before, in this case any peripheral cycle must be a face boundary. Our goal here will be to prove the following.

Theorem 1.5. *Every strongly 2-edge-connected 2-regular digraph has at least two peripheral cycles through every edge.*

Corollary 1.6. *Every strongly 2-edge-connected 2-regular digraph which has a special embedding in the plane has a unique special embedding in the plane.*

Proof of Theorem 1.5. Let $e = (u, v)$ be an edge of G. Our first goal will be to find one peripheral cycle through e. To do this, we choose a directed path P from v to u so as to lexicographically maximize the sizes of the components of $G' = G - (E(P) \cup \{e\})$. That is, we choose the path P so that the largest component of G' is as large as possible, and subject to this the second largest is as large as possible, and so on.

Suppose (for a contradiction) that G' has components $G_1, G_2, \ldots, G_k$ with $k > 1$ where G_k is a smallest component. Let P' be the minimal directed path of P which contains all vertices of G_k and suppose the start of P' is the vertex x and the last vertex is y. By construction, G_k must contain both x and y. Furthermore, since G_k is Eulerian, both of these vertices have indegree and outdegree equal to one in G_k. If there is a component G_i with $i < k$ which contains a vertex in the interior of P', then we may choose a directed path P'' in G_k from x to y (since G_k is Eulerian, it is automatically strongly connected). Now we get a contradiction, since we can reroute the original path along P'' instead of P' and get a new path which improves our lexicographic ordering. Thus, all vertices in the interior of P' must also be in G_k. However, in this case $G_k \cup P'$ is an induced subgraph which is separated from the rest of the graph by just two edges, and we have a contradiction to the strong 2-edge-connectivity. It follows that $k = 1$, so the cycle $P \cup \{e\}$ is indeed peripheral.

Since the cycle $P \cup \{e\}$ is peripheral, there exists a directed path Q with $E(Q) \cap E(P) = \emptyset$ from v to u. Among all such directed paths Q we choose one so that the unique component of $G - (E(Q) \cup \{e\})$ which contains P is as large as possible, and subject to that we lexicographically maximize the sizes of the remaining components. By the same argument as above, this choice will result in another peripheral cycle. □

2 Flows in bidirected graphs

2.1 Colouring-flow duality in the plane

We begin with a lovely observation due to Tutte which opened the study of this field. Before stating it we will need to introduce some basic terminology.

Definition: Let Γ be an abelian group (written additively), and let $G = (V, E)$ be a directed graph. We define a function $\phi : E \to \Gamma$ to be a *flow* if the following condition (called the Kirchoff rule) is satisfied at every vertex $v \in V$

$$\sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e) = 0.$$

So, in words, a function is a flow, if at every vertex v, the sum of the values on the incoming edges is equal to the sum of the values on the outgoing edges.

We say that a flow is a *k-flow* when $\Gamma = \mathbb{Z}$ and $|\phi(e)| < k$ for every $e \in E$; we call ϕ *nowhere-zero* if $\phi(e) \neq 0$ for every $e \in E$. Note that if we have a flow, then we can reverse an edge and change its value to $-\phi(e)$ and this preserves the Kirchoff rule, so we still have a flow. This also preserves the properties of k-flow and nowhere-zero flow. Accordingly, we will say that an undirected graph has a *nowhere-zero Γ flow* or *nowehere-zero k-flow* if some (and thus every) orientation of it has this property.

Theorem 2.1 (Tutte)**.** *If G and G^* are dual planar graphs, then G^* has a proper k-colouring if and only if G has a nowhere-zero k-flow.*

Proof of the "only if" direction. (see next section for the "if" direction) Let V^* be the set of vertices of G^* and also the set of faces of G and suppose that $g : V^* \to \{0, 1, \ldots, k-1\}$ is a proper k-colouring. Now, orient the edges of G arbitrarily and assign each edge e of G a value $\phi(e)$ according to the rule that $\phi(e) = g(a) - g(b)$ where a is the face to the left of the directed edge e (when it is oriented upward) and b is the face to the right. To check that this is a flow, consider a vertex v and suppose first that all edges are directed away from v. In this case, the Kirchoff rule will be satisfied at v because within this sum each face a incident with v contributes $g(a) - g(a) = 0$. If we flip the direction of an edge, this flips its sign, so the Kirchoff rule will still be satisfied. Since our colouring was proper, the resulting function ϕ is indeed a nowhere-zero k-flow, as desired. ☐

Note that a planar graph with a loop edge does not have any proper colouring, so it's dual does not have any nowhere-zero k-flow. More generally, any graph with a cut-edge will not have a nowhere-zero Γ-flow for any (abelian) group Γ. To see this, just sum the Kirchoff rule over all vertices in one component of $G - e$ for a cut-edge e. Since we have a flow, this sum must be zero, but all terms in this sum apart from $\phi(e)$ cancel, so it gives $\phi(e) = 0$. Based on the above theorem connecting flows and colourings, Tutte made three remarkable conjectures concerning the existence of nowhere-zero flows, all of which are still open despite considerable efforts.

Conjecture 2.2 (Tutte)**.** *Let G be a graph without a cut-edge.*

1. *Then G has a nowhere-zero 5-flow.*
2. *If G has no Petersen minor, it has a nowhere-zero 4-flow.*
3. *If G is 4-edge-connected, it has a nowhere-zero 3-flow.*

The first of these conjectures, the 5-flow conjecture, holds true for planar graphs by the 5-colour theorem. The Petersen graph does not have a nowhere-zero 4-flow, so if it is true, the 5-flow conjecture would be best possible. Seymour proved that every graph without a cut-edge has a nowhere-zero 6-flow, and this result will be of significance for our forthcoming discussion.

The 4-flow conjecture when restricted to cubic graphs is equivalent to the statement that every cubic graph with no cut-edge and no Petersen minor has a 3-edge colouring. This was proved by Robertson, Sanders, Seymour and Thomas, but little more is known in general. The last of these conjectures holds true for planar graphs since it dualizes to the statement that every triangle free planar graph is 3-colourable—which was proved by Grötzsch.

Before leaving this section let us close with another easy observation and another useful theorem of Tutte. Observe that our proof of the "only if" direction of Theorem 2.1 actually gives a somewhat more general result. If instead of choosing a k-colouring using the colours $\{0, 1, \ldots, k-1\}$ we had instead chosen Γ to be an abelian group of order k and chosen $g : E \to \Gamma$ to be our colouring, then the construction would have resulted in a nowhere-zero Γ flow. So, a k-colouring of the dual naturally gives us either a nowhere-zero k-flow or a nowhere-zero Γ-flow in the original graph G. The following theorem shows that this phenomena holds true in a more general setting.

Theorem 2.3 (Tutte)**.** *For every positive integer k and graph G, the following are equivalent.*

1. *G has a nowhere-zero k-flow.*
2. *G has a nowhere-zero Γ flow for some group Γ with $|\Gamma| = k$.*
3. *G has a nowhere-zero Γ flow for every group Γ with $|\Gamma| = k$.*

The utility of this result becomes immediately apparent when one starts working with flows. The reason is that it is easy to modify a Γ-flow to get another Γ-flow (for instance by adding a constant value to all edges on a directed cycle), but it is generally difficult to modify a k-flow to get another k-flow.

2.2 Duality for orientable surfaces

Let's consider a directed graph $G = (V, E)$ which is embedded in an orientable surface. Let $\phi : E \to \Gamma$ be a flow on G. Now we may construct the dual graph $G^* = (V^*, E^*)$ and orient its edges so that whenever $e \in E$ corresponds to $e^* \in E^*$, the edge e^* crosses left to right over e. Now consider the function $\phi^* : E^* \to \Gamma$ given by the rule $\phi^*(e^*) = \phi(e)$. For a walk W in G^* with edge sequence $e_1^*, \ldots, e_m^*$ we define the *height* of this walk to be

$$\phi^*(W) = \sum_{i=1}^{m} \epsilon_i \phi(e_i^*)$$

where $\epsilon_i = 1$ if e_i^* is traversed forward, and $\epsilon_i = -1$ if e_i^* is traversed backward. With this notation in place, we see that the Kirchoff rule for a vertex v in the original graph is precisely equivalent to the statement that the closed walk bounding the face of G^* corresponding to v has height 0. So, our flow ϕ dualizes to give a function ϕ^* with the property that every facial walk has height 0. This is an important concept, so let's pause to define it.

Definition: For an embedded directed graph G and a function $\psi : E(G) \to \Gamma$, we say that ψ is a *local-tension* if the height of every facial walk is 0. We say that ψ is a *tension* if every closed walk has height 0.

Just as with flows, we will call a (local) tension ψ *nowhere-zero* if $\psi(e) \neq 0$ for every $e \in E(G)$ and we call ψ a *k-(local) tension* if $\Gamma = \mathbb{Z}$ and $|\psi(e)| < k$ for every $e \in E(G)$. Also just like flows, we can reverse the direction of an edge and multiply its value by -1 to obtain a new (local) tension, so the question of when a directed graph has a nowhere-zero (local) tension depends only on the underlying graph and not the orientation. The following key result shows that nowhere-zero tensions are essentially the same as colourings.

Proposition 2.4. *A graph G has a nowhere-zero Γ-tension if and only if it has a proper $|\Gamma|$-colouring.*

Sketch of proof. For the "if" direction, choose a Γ-colouring of G given by $g : V(G) \to \Gamma$. Then orient the edges of G arbitrarily and assign the value $\psi(e) = g(v) - g(u)$ if e is an edge directed from v to u. It is straightforward to check that this gives a nowhere-zero tension.

For the"only if" direction choose a nowhere-zero tension $\psi : E(G) \to \Gamma$ and then fix a base point $u \in V(G)$. Now for every vertex $v \in V(G)$ choose a walk W_v from u to v and define $g(v) = \psi(W_v)$. It follows from the assumption that ψ is a tension that the value $g(v)$ does not depend on the choice of W_v. Moreover, the assumption that ψ was nowhere-zero means that the resulting function g is a proper colouring. □

Assume that we have a tension ψ of an embedded graph G, and assume that every face in this embedding is a disc. If W is a closed walk in the graph which forms a contractible curve in the surface, then we may deform W to a trivial walk by rerouting along faces one at a time. It follows from this that $\psi(W) = 0$. More generally, let us fix a base point $u \in V(G)$ and consider two closed walks W and W' starting and ending at u. If W and W' are homotopic, then by the same argument, we deduce that $\psi(W) = \psi(W')$. This leads us to the following key property.

Proposition 2.5. *Let G be a directed graph embedded in a surface $\mathcal{S}$. If $\psi : E(G) \to \Gamma$ is a local-tension, then ψ induces a group homomorphism from $\pi_1(S) \to \Gamma$. This homomorphism is trivial if and only if ψ is a tension.*

With this, we can now return to prove the other part of our first theorem.

Proof of "if" direction of Theorem 2.1. By assumption, the graph G has a nowhere-zero k-flow. So, by Theorem 2.3 we may orient G and choose a nowhere-zero $\mathbb{Z}_k$ flow ϕ. Let G^* be the oriented dual (as above) and define $\phi^*(e^*) = \phi(e)$ for every edge $e^* \in E(G^*)$. Then ϕ^* is a nowhere-zero $\mathbb{Z}_k$-local tension. However, since the homotopy group of the plane is trivial, the above proposition implies that ϕ^* is actually a tension. Thus by Proposition 2.4 the graph G^* has a proper k-colouring. □

2.3 Duality for nonorientable surfaces

Now we shall start off with a directed graph G which is embedded in a non-orientable surface, and a local tension $\psi : E(G) \to \Gamma$. Our aim is to translate the local-tension property into a kind of flow in the dual graph. However, since our surface is not orientable, there is no obvious orientation of the dual to use. In fact, we will need a more complicated notion. A *bidirected graph* is a graph in which every edge has two arrowheads, one associated with each endpoint. Just as with usual directed graphs, these arrowheads may be directed either toward or away from this endpoint.

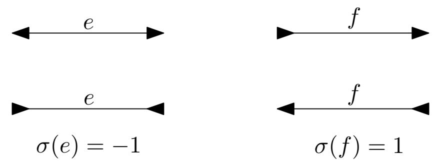

Figure 4: edge types

We assume (as usual) that every face of the embedded graph G is a disc, and we will think of each of these discs as equipped with a local notion of clockwise. (This is one of the many ways of working with nonorientable surface embeddings.) Let G^* be the dual graph, and consider the face of G which is associated to some vertex $v^* \in V$. We have chosen a clockwise orientation of this face, and we let W_{v^*} be a facial walk which traverses this face clockwise. Now, by assumption we have $\psi(W_{v^*}) = 0$ and we shall translate this into a flow type condition at the vertex v^* in the dual. To do so, just mark each edge e^* of G^* which is incident with v^* with an arrowhead directed to v^* if the corresponding edge e of G is forward in W_{v^*} and with an arrowhead the opposite direction if it is backward. This immediately translates the property $\psi(W_{v^*}) = 0$ into the Kirchoff rule being satisfied at v^*. However, if we do this at every vertex of the dual, we will in general end up with a bidirected orientation of this dual graph

Following the above process and giving the dual graph G^* a bidirected orientation results in the duality we want. Namely, we will have that our local tension ψ of G translates into a flow ψ^* of the dual (bidirected) graph G^*. So, just as we could use nowhere-zero flows in ordinary digraphs to construct local-tensions on orientable surfaces, we can now use nowhere-zero flows in bidirected graphs to construct local-tension on non-orientable surfaces. One might have hoped that the analogue of Tutte's 5-flow conjecture would still hold true for bidirected graphs, that is that every bidirected graph without the obvious obstruction has a nowhere-zero 5-flow, but this is not true. To see why, consider the dual graphs K_6 and Petersen embedded in the projective plane. Direct the edges of K_6 and use the above procedure to give Petersen a bidirected orientation. Now consider any local tension $\phi : E(K_6) \to \mathbb{Z}$ of K_6. By Proposition 2.5 this local tension induces a group homomorphism from the fundamental group of our surface, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ to the group $\mathbb{Z}$. Since this must be trivial, we deduce that ϕ must be a tension. It follows that this embedded K_6

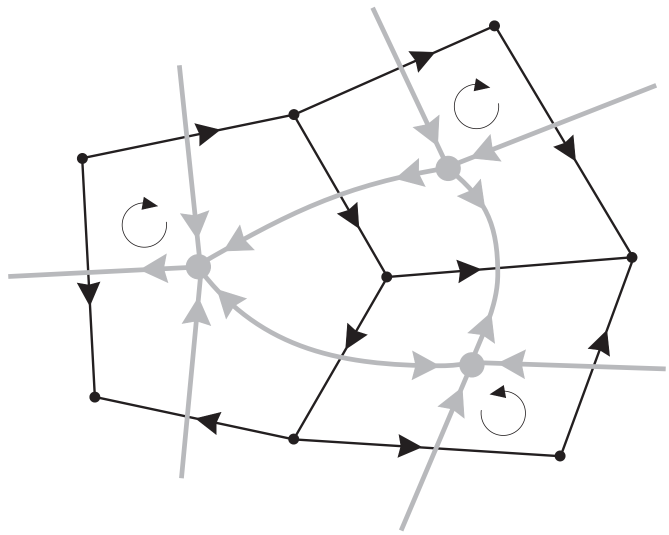

Figure 5: duality

does not have a 5-local-tension, and then by duality the associated bidirected Petersen does not have a nowhere-zero 5-flow. Bouchet conjectured that this was the most extreme example.

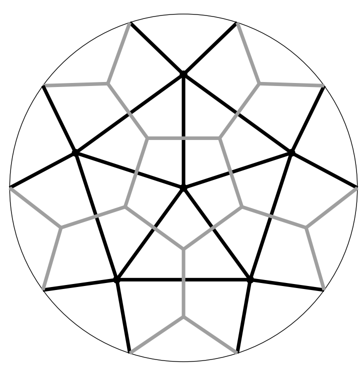

Figure 6: Petersen and K_6 in the projective plane

Conjecture 2.6 (Bouchet)**.** *Every bidirected graph with a nowhere-zero* $\mathbb{Z}$*-flow has a nowhere-zero* 6*-flow.*

Bouchet proved that graphs with nowhere-zero $\mathbb{Z}$-flows have nowhere-zero 216-flows. This was improved to 30-flows by Fouquet and independently by Zyka. We have shown that such graphs have nowhere-zero 12-flows.

3 Average degree of graph powers

This article will eventually turn to a very basic question in graph theory. However, we shall begin with our motivation, which comes from the world of additive number theory.

3.1 Groups

Let Γ be an abelian group (written additively). For two sets $A, B \subseteq \Gamma$ we define

$$A + B = \{a + b \mid \ a \in A \text{ and } b \in B\}$$

and we call such a set a sumset. One of the central problems in additive combinatorics is understanding the structure of finite sets A for which the sumset $A + A$ is small. Let's begin with an easy case where our group is the integers.

Observation 3.1. *If $A \subseteq \mathbb{Z}$ is finite and nonempty, then $|A + A| \geq 2|A| - 1$. Moreover, if this bound is met with equality, then A is an arithmetic progression.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 \ldots < a_n$. Then we may exhibit $2n - 1$ distinct members of the sumset $A + A$ as follows

$$a_1 + a_1 < a_1 + a_2 < \ldots a_1 + a_n < a_2 + a_n \ldots < a_n + a_n.$$

This gives us the desired bound.

Now we investigate the case where our set A hits this bound with equality. Generalizing the above procedure, we can construct a list of $2n - 1$ distinct members of $A + A$ by starting with $a_1 + a_1$ and moving to $a_n + a_n$ by increasing the index of either the left or right term by one at each stage. If $|A+A| = 2n-1$ then we must get the same list of integers however we do this. Since the k^{th} term in such a list could be either $a_1 + a_{k+1}$ or $a_2 + a_k$ it follows that every $1 \leq k < n$ must satisfy $a_2 - a_1 = a_{k+1} - a_k$. Therefore, A is an arithmetic progression. $\square$

Now we shall turn our attention from the integers to the group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ in the case when p is prime. Here there is a new reason why the set $A + A$ might be small relative to A, namely A could be all, or almost all of the group. The following famous theorem asserts that in this group we either get the same bound we had for the integers, or $A + A = \mathbb{Z}_p$.[1]

Theorem 3.2 (Cauchy-Davenport)**.** *Let p be prime and let $A \subseteq \mathbb{Z}_p$ be nonempty. Then we have*

$$|A + A| \geq \min\{p, 2|A| - 1\}.$$

There is an also a characterization of the sets $A \subseteq \mathbb{Z}_p$ for which $|A+A| < 2|A|$ due to Vosper.[2]

[1] In fact, this theorem has a more general form which involves sumsets of the form $A + B$.

[2] As with Cauchy-Davenport, Vosper's theorem applies more generally to sets A, B with $|A + B| < |A| + |B|$.

Theorem 3.3 (Vosper). *Let p be prime, let $A \subseteq \mathbb{Z}_p$ is nonempty, and assume $|A + A| < 2|A|$. Then one of the following holds:*

1. *A is an arithmetic progression.*
2. *$|A + A| \geq p - 1$.*

There are similar results which hold in more general contexts, such as the following result which we do not state precisely. Here we have switched to multiplicative notation for the group Γ since this is the common convention when working with groups which are permitted to be nonabelian. So $A \cdot A = \{a \cdot a' \mid a, a' \in A\}$.

Theorem 3.4 (D.). *Let A be a finite generating set of the multiplicative group Γ and assume $1 \in A$. If $|A \cdot A| < 2|A|$ then one of the following holds*

1. *Γ has a normal subgroup K so that Γ/K is either cyclic or dihedral.*
2. *There exists a proper coset K so that $\Gamma \setminus K \subseteq A \cdot A$.*

In fact, there are very wide sweeping generalizations of these results which concern sets A for which $|A \cdot A| < c|A|$ for a fixed constant c. There are structure theorems here due to Green-Ruzsa for abelian groups and due to Breulliard-Green-Tao for arbitrary groups which yield profound insights into the nature of these groups. We will not pursue this direction, but shall instead try to take some of the behaviour we see here and prove that similar things happen without all of the structure of a group.

3.2 Graphs

Assume now that Γ is a multiplicative group and let $A \subseteq \Gamma$. The Cayley Graph $Cayley(\Gamma, A)$ is a directed graph with vertex set Γ and an edge (x, y) whenever $y \in xA$. So, in words, there is an edge from x to y if you can get from x to y my multiplying on the right by some element in A. Let $g \in \Gamma$ and consider the bijection of Γ given by the rule $x \to gx$. It follows immediately from our definition that this map sends directed edges to directed edges, so this gives an automorphism of our digraph. Since there is such an automorphism sending any vertex to any to any other vertex, every Cayley graph is vertex transitive.

One convenient property of Cayley graphs is that they permit us to analyze questions about small product sets using graphs. Indeed, for $Cayley(\Gamma, A)$ the size of A is precisely the degree of this regular digraph, and the size of the set $A \cdot A$ is precisely the number of vertices reachable from a given fixed vertex x by taking two (directed) steps. This gives us hope of following the theme of the previous section in a more general setting of digraphs instead of Cayley graphs. There are many nice questions in this realm which are unsolved. Here is one of my favourite.

Conjecture 3.5. *Let G be a simple d-regular digraph (all indegrees and outdegrees equal to d) with no directed cycles of length* 1 *or* 2*. Then there exists a* *vertex $x \in V(G)$ so that x can reach at least $2d$ vertices by a forward path of length 1 or 2.*

If true the above would resolve a very special case of the following very famous unsolved problem. (Namely the case when G is regular and $k = 3$).

Conjecture 3.6 (Caccetta-Häggkvist)**.** *Let k be a positive integer and let G be a simple n-vertex digraph. If every vertex in G has outdegree at least n/k, then G has a directed cycle of length at most k.*

As is common in graph theory, digraphs are awfully tricky and undirected graphs behave better. The following theorem is a related success for undirected graphs. Here the graph G^k denotes the simple graph with vertex set $V(G)$ and two vertices u, v adjacent in G^k if they have distance at most k in G.

Theorem 3.7 (D., Thomassé)**.** *If G is a simple connected graph of minimum degree d and diameter at least 3, then the average degree of G^3 is at least $\frac{7}{4}d$.*

A proof can be found in our paper on arXiv [2].

References

[1] M. DeVos. Flows on bidirected graphs. `https://arxiv.org/abs/1310.8406`.

[2] M. DeVos and S. Thomassé. Edge growth in graph cubes. `https://arxiv.org/abs/1009.0343`

Classical graph properties and graph parameters and their definability in SOL

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Editors' note: The text below is an adaptation and abridgement of the slides that supported the lectures.[1] The lectures were based on joint work with Tomer Kotek.

Course outline

LECTURE 01: Friday, Oct 10, 2014, 14:00-15:40, Prague Lecture 1,
A landscape of graph parameters and graph polynomials. Comparing graph parameters. Towards a general theory.

LECTURE 02: Thursday, Oct 16, 2014, 12:20-14:00 Prague Lecture 2,
Why is the chromatic polynomial a polynomial? Where do graph polynomials occur naturally? Definability of graph properties and graph polynomials in a fragment of second-order logic.

LECTURE 03: Thursday, Oct 16, 2014, 14:30-16:00 Prague Lecture 3,
Connection matrices for graph parameters. When do connection matrices of graph parameters have finite rank? Connection matrices for graph parameters definable in fragments of second-order logic. The finite rank theorem. Using connections matrices to prove non-definability.

There is a **LECTURE 00** on second-order logic (SOL) and its fragments (background, not lectured), LOGICS, with 14 slides. The slides can be found at `http://www.cs.technion.ac.il/~janos/COURSES/Prague-2014/P-logics.pdf`.

1 A landscape of graph parameters and graph polynomials

- Introducing graph polynomials
- The chromatic polynomial
- The characteristic polynomial

[1] An extended set of slides can be found at the FMT-2012 program page:
Program:`http://www.lsv.ens-cachan.fr/Events/fmt2012/program.php`
Slides:`http://www.lsv.ens-cachan.fr/Events/fmt2012/SLIDES/janosmakowsky.pdf` and at `http://www.cs.technion.ac.il/~janos/COURSES/Prague-2014/`

- The matching polynomials
- Multivariate graph polynomials: the Tutte polynomial
- Complete graph invariants
- Comparing graph invariants: getting started
- Comparing graph invariants: towards a general theory
- Semantic versus syntactic properties of graph parameters

1.1 Introducing graph polynomials

Let $\mathcal{DG}$ be the class of finite graphs $\langle V(G), E(G)\rangle$ where $V = V(G)$ is a finite set and $E = E(G) \subseteq V^2$ is a set of (directed edges). $G \in \mathcal{DG}$ is called a directed graph. Let $\mathcal{G}$ be the class of finite graphs, i.e. where E is symmetric.

For $G_1, G_2 \in \mathcal{DG}$ $f : G_1 \to G_2$ is an **isomorphism** if

1. f is a bijection, and
2. For $u, v \in V(G_1)$ we have
   $(u, v) \in E(G_1)$ iff $(f(u), f(v)) \in E(G_2)$.

G_1 and G_2 are isomorphic, denoted by $G_1 \simeq G_2$, if there is an isomorphism $f : G_1 \to G_2$.

Let $\mathcal{R}$ denote a ring. For example: $\mathcal{B}_2$ the two-element boolean ring, $\mathbb{Z}_2$ the two element field, $\mathbb{Z}$ the ring of integers, $\mathbb{Z}[X]$ the polynomial ring over the integers with one indeterminate, or $\mathbb{R}$ the ring of real numbers.

Definition 1.1. Let $\mathcal{R}$ a ring, $\mathcal{G}$ the class of finite graphs. A function

$$f : \mathcal{G} \to \mathcal{R}$$

is a graph invariant if for any two isomorphic graphs G_1, G_2 we have $f(G_1) = f(G_2)$.

Boolean graph invariants. Here the ring is $\mathcal{B}_2$, or any ring $\mathcal{R}$, but the values of the invariant are either 0 or 1.

- Connectedness
- Regular, or regular of degree r.
- Any First-Order-expressible graph property.
- Any Second-Order-expressible graph property.
- Belonging to any specific class of graphs closed under isomorphisms.
- There are continuum-many boolean graph invariants.

Numeric graph invariants. Here the ring is $\mathbb{Z}$.

- The cardinality of $V(G)$ or $E(G)$.
- The number of connected components of G, usually denoted by $k(G)$.
- The coloring (chromatic) number of G.
- The size of the maximal clique (independent set).
- The diameter of G.
- The radius of G.
- The minimum length of a cycle in G, if it exists, called the girth of G.

Graph polynomials. Here the ring is $\mathbb{Z}[X]$.

The graph polynomial $p(G, X)$ gives for each value of X a graph invariant, hence it encodes a possibly infinite family of graph invariants. The study of graph polynomials has a long history concentrating on particular polynomials.

The *classic* and very readable book is [2].

1.2 The chromatic polynomial

Let $\chi(G, X)$ denote the number of vertex colorings of G with X colors. We shall prove that $\chi(G, X)$ is a polynomial in X, called the *chromatic polynomial of* G.

The chromatic polynomial was first introduced by G.D. Birkhoff in 1912. It led to a very rich theory, although it was introduced in a (failed) attempt to prove the 4-color conjecture.

The most comprehensive monograph about the chromatic polynomial is [8].

What can we do with a graph polynomial?

- Study its zeros.
- Interpret its coefficients in various normal forms.
- Interpret its evaluations.
- Study graphs uniquely determined by the polynomial.
- Study graph classes having the same graph polynomial.
- Study its strength as a graph invariant in the sense of distinguishing non-isomorphic graphs.

Digression: Typical theorems about the chromatic polynomial.

Theorem 1.2 (G. Birkhoff, 1912)**.** *$\chi(G, X)$ is a polynomial in X of degree $|V(G)|$.*

Proof. Let $e = (a, b)$ be an edge of the graph G. $G - e$ and G/e are obtained from G by deleting, respectively contracting the edge e.
We use induction on $|E(G)|$.

- First we observe that for disjoint unions $G = G_1 \sqcup G_2$
  we have $\chi(G, X) = \chi(G_1, X) \cdot \chi(G_2, X)$.
- For n isolated points $\bar{K_n}$ we have $\chi(\bar{K_n}, X) = X^n$.
- $\chi_{a \neq b}(G, X)$ is the number of X-colorings of G with a and b having different colors.
- $\chi_{a=b}(G, X)$ is the number of X-colorings of G with a and b having the same color.
- $\chi(G - e, X) = \chi_{a \neq b}(G - e, X) + \chi_{a=b}(G - e, X) = \chi(G, X) + \chi(G/e, X)$
- $\chi(G, X) = \chi(G - e, X) - \chi(G/e, X)$

$\square$

Normal forms of $\chi(G, X)$, I. As $\chi(G, X)$ is a polynomial we can write it as

$$\chi(G, X) = \sum_{i=0}^{|V(G)|} b_i(G) X^i.$$

For the disjoint union we noted that

Proposition 1.3. $\chi(G_1 \sqcup G_2, X) = \chi(G_1, X) \cdot \chi(G_2, X)$.

Normal forms of $\chi(G, X)$, II. We define $X_{(i)} = X \cdot (X-1) \cdot \ldots \cdot (X-i+1)$. We write

$$\chi(G, X) = \sum_{i=0}^{|V(G)|} c_i(G) X_{(i)}$$

We define an operation $\circ$ on the $X_{(i)}$ by $X_{(i)} \circ X_{(j)} = X_{(i+j)}$ and extend it linearly to polynomials in $X_{(i)}$.

The join of two graphs G_1, G_2, $G_1 + G_2$, is obtained by taking the disjoint union and adding all the edges between $V(G_1)$ and $V(G_2)$.

Theorem 1.4.

$$\chi(G_1 + G_2, X) = \left(\sum_{i=0}^{|V(G)|} c_i(G_1) X_{(i)} \circ \sum_{i=0}^{|V(G)|} c_i(G_2) X_{(i)} \right)$$

Trees and tree-width.

- For trees T with n vertices we have $\chi(T, X) = X \cdot (X-1)^{n-1}$. In particular, any two trees on n vertices have the same chromatic polynomial.
- (R. Read, 1968)
  Conversely, for G a simple graph, if $\chi(G, X) = X \cdot (X-1)^{n-1}$ then G is a tree.
- (C. Thomassen, 1997)
  If G has tree-width $k \geq 2$ then for every real number $a > k$ we have $\chi(G, a) \neq 0$.
- (B. Courcelle, J.A. Makowsky, U. Rotics, 2000)
  For graphs G with tree-width at most k the polynomial $\chi(G, X)$ can be computed in polynomial time.
- (J.A. Makowsky, U. Rotics, 2005)
  For graphs G with clique-width at most k the polynomial $\chi(G, X)$ can be computed in polynomial time.

Planar graphs and the chromatic polynomial.

Theorem 1.5 (P.J. Heawood, 1890)**.** *Every planar graph is 5-colorable, i.e., $\chi(G, 5) \neq 0$ for G planar.*

Theorem 1.6 (G. Birkhoff and D. Lewis, 1946)**.** *$\chi(G, a) \neq 0$ for G planar and $a \in \mathbb{R}, a \geq 5$.*

Note that Theorem 1.6 is much stronger than Heawood's 5-color theorem.

Theorem 1.7 (K. Appel and W. Haken, 1977)**.** *Every planar graph is 4-colorable. $\chi(G, 4) \neq 0$ for G planar.*

Problem 1.8. *Find an analytic proof of the 4-color theorem.*

Conjecture 1.9 (G. Birkhoff and D. Lewis, 1946)**.** *For G planar, there are no real roots of $\chi(G, a)$ for $4 \leq a \leq 5$.*

Real roots of $\chi(G, X)$. We note that $\chi(G, 0) = 0$ for any graph with at least one vertex, and $\chi(G, 1) = 0$ for any graph with at least one edge.

Theorem 1.10 (D. Woodall, 1977)**.** *Let G be any graph.*

- *There are no negative real roots of $\chi(G, X)$.*
- *There are no real roots of $\chi(G, X)$ in the open interval $(0, 1)$.*

Theorem 1.11 (B. Jackson, 1993)**.**

- *There are no real roots of $\chi(G, X)$ in the semi-open interval $(1, \frac{32}{27}]$.*
- *For any $\epsilon > 0$ there is a graph G_ϵ such that $\chi(G_\epsilon, X)$ has a root in $(\frac{32}{27}, \frac{32}{27} + \epsilon)$.*

Theorem 1.12 (S. Thomassen, 1997)**.** *For any real numbers a_1, a_2 with $\frac{32}{27} \leq a_1 < a_2$ there exists a graph G such that $\chi(G, X)$ has a root in (a_1, a_2).*

Other counting interpretations: acyclic orientations. An *orientation* of a graph G is a function which for every edge $e = (a, b)$ selects a source value $s(e) \in \{a, b\}$ An orientation is *acyclic*, if there are no oriented cycles.

Theorem 1.13 (R.P. Stanley, 1993)**.** *The number of acyclic orientations of a graph G is given by the absolute value $|\chi(G, -1)|$.*

Subgraph expansions. Let G be a graph with $k(G)$ connected components. Let $S \subset E(G)$ and denote by $\langle S \rangle$ the subgraph generated by S in G.

- The *rank* $r(G)$ is defined as $r(G) = |V(G)| - k(G)$.
- The *corank* $s(G)$ is defined as $s(G) = |E(G)| - |V(G)| + k(G)$.
- The *rank polynomial* of a graph is defined by

$$R(G; X, Y) = \sum_{S \subseteq E(G)} X^{r(\langle S \rangle)} Y^{s(\langle S \rangle)}$$

Theorem 1.14 (H. Whitney, 1932)**.**

1. $\chi(G, X) = \sum_{S \subseteq E(G)} (-1)^{|S|} X^{|V(G)| - r(\langle S \rangle)}$.
2. $\chi(G, X) = X^{|V|} R(G, -X^{-1}, -1)$.

1.3 The characteristic polynomial

- Let $V = [n]$ and let A_G be the (symmetric) adjacency matrix of G with $(A)_{j,i} = (A)_{i,j} = 1$ iff there is an edge between vertex i and vertex j.
- We denote by $char(G, X)$ the polynomial

$$\det(X \cdot \mathbf{1} - A) \tag{1}$$

  $char(G, X)$ is a graph invariant and a polynomial in X, called the *characteristic polynomial of G*.

- The set of roots of $char(G, X)$ (with multiplicities) are the eigenvalues of A_G, and are called the *spectrum of the graph* G.

The characteristic polynomial and the spectrum of a graph were first studied in the 1950s by: T.H. Wei 1952, L.M. Lihtenbaum 1956, *L. Collatz and U. Sinogowitz* 1957, H. Sachs 1964, H.J. Hoffman 1969.

The characteristic polynomial: literature. The characteristic polynomial and spectra of graphs have a very rich literature with important applications in chemistry under the name *Hückel theory.* See [2, 6, 7, 22].

Digression: typical theorems about the characteristic polynomial.

Coefficients of $char(G, X)$. For a graph G on n vertices, we write

$$char(G, X) = \sum_{i=0}^{n} c_i(G) \cdot X^{n-i}$$

Proposition 1.15.
1. $c_0 = 1$
2. $c_1 = 0$
3. $-c_2 = |E(G)|$ *is the number of edges of* G.
4. $-c_3$ *is twice the number of triangles of* G.

Eigenvalues of G. As in linear algebra, the zeros of $char(G, X)$ are called *eigenvalues of the matrix* A_G, or *eigenvalues of the graph* G,

Proposition 1.16.
1. *All the eigenvalues of* G *are real.*
2. *If* G *is connected, the largest eigenvalue of* G *has multiplicity* 1.
3. *If* G *is connected and of diameter at least* d, *the* G *has at least* $d + 1$ *distinct zeros.*
4. *The complete graph is the only connected graph with exactly two distinct eigenvalues,* $char(K_n, X) = (X + 1)^{n-1}(X - n + 1)$.
5. *Let* $\Lambda(G)$ *be the largest eigenvalue of* G. G *is bipartite iff* $-\Lambda(G)$ *is also an eigenvalue of* G.

Proposition 1.17. *Let* G *be a regular graph of degree* r. *Then*
1. r *is an eigenvalue of* G
2. *If* G *is connected, then the multiplicity of* r *is* 1.
3. *For any eigenvalue* λ *of* G *we have* $|\lambda| \leq r$.
4. *The multiplicity of the eigenvalue* r *is the number of connected components of* G.

$\lambda(G)$ denotes the smallest eigenvalue of G. $\lambda_2(G)$ denotes the second largest eigenvalue of G. $\Lambda(G)$ denotes the largest eigenvalue of G.

Proposition 1.18.
1. *If* H *is an induced subgraph of* G, *then* $\lambda(H) \leq \lambda(G)$.

2. *If H is an induced subgraph of G, then $\Lambda(H) \leq \Lambda(G)$. If H is a proper induced subgraph, then $\Lambda(H) < \Lambda(G)$.*
3. *For no graph G is $\lambda(G) \in (-1, 0)$.*
4. *Let G have at least two vertices. $\lambda(G) = -1$ iff G is a complete graph.*
5. *For no graph G is $\lambda(G) \in (-\sqrt{2}, -1)$.*
6. *(J. Smith, 1970) $\lambda_2(G) \leq 0$ iff G is a complete multipartite graph.*

1.3.1 The acyclic or matching defect polynomial

We denote by $m_k(G)$ the number of k-matchings of a graph G, with $m_0(G) = 1$ by convention. For a graph G on n vertices, the polynomial

$$dm(G, X) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) X^{n-2k} \tag{2}$$

is called the *acyclic polynomial of G* and also the *reference polynomial* or *matching defect polynomial*.

The acyclic polynomial has important applications in chemistry (Hückel theory again) and and the molecular physics of ferromagnetism. It was first studied in the 1970s (Heilman and Lieb, Kunz). See [18, 22, 10].

1.4 The matching (generating) polynomial

The polynomial

$$gm(G, X) = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k(G) X^k \tag{3}$$

is called the *matching polynomial of G* or the *matching generating polynomial of G*. It is easy to verify the identity

$$dm(G, X) = X^n gm(G, (-X^{-2})). \tag{4}$$

1.5 Multivariate graph polynomials

Inspired by H. Whitney's work (1932), W.T. Tutte (1947, 1954) investigated generalizations of the chromatic polynomial to a polynomial in two variables, which he called the **dichromatic polynomial**, but now called the *Tutte polynomial*, $T(G, X, Y)$.

The Tutte polynomial and its many generalizations became prominent due to its many combinatorial interpretations in fields outside graph theory:

- Knot theory (via the Jones polynomial and its relatives)
- Statistical mechanics
- Quantum theory and quantum computing
- Chemistry

1.6 The Tutte polynomial

Let $G = (V, E)$ be a graph, and for $A \subseteq E$, let $G_A = (V, A)$ be a spanning subgraph. The rank $r(G; A)$ is defined as $|V(G)| - k(G_A)$.

The *Tutte polynomial of* G is defined as

$$T(G; X, Y) = \sum_{A \subseteq E} (X-1)^{r(G;E)-r(G;A)} \cdot (Y-1)^{|A|-r(G;A)} \tag{5}$$

This looks confusing and innocent at the same time.

The fascination with the Tutte polynomial. The Tutte polynomial is like a **magician's hat** with **rabbits, birds and other surprises** coming out. Easy manipulations produce various combinatorial counting functions. We have, at first glance surprisingly, the following

- $T(G, 1, 1)$ counts the number of spanning trees of G.
- $T(G, 2, 1)$ counts the number of forests of G.
- $T(G, 2, 0)$ counts the number of acyclic orientations of G.
- The chromatic polynomial is given by

$$\chi(G, X) = (-1)^{r(G;E)} X^{k(G)} T(G; 1 - X, 0)$$

- The reliability polynomial and the flow polynomial can also be derived with similar formulas.

1.7 Complete graph polynomials

A graph invariant f is *graph-complete* if for any two graphs G_1, G_2 with $f(G_1) = f(G_2)$ we have also $G_1 \simeq G_2$.

Are there complete graph polynomials? The following is a graph-complete graph invariant.

- Let $X_{i,j}$ and Y be indeterminates. For a graph $\langle V, E\rangle$ with $V = [n]$ we put

$$Compl(G, Y, \bar{X}) = Y^{|V|} \cdot \left(\sum_{\sigma \in \mathfrak{S}_n} \prod_{(i,j) \in E} X_{\sigma(i),\sigma(j)} \right)$$

Here $\mathfrak{S}_n$ is the permutation group of $[n]$.

Challenge: Find a polynomial in a constant finite number of indeterminates which is a graph-complete graph invariant.

An "unnatural" graph-complete invariant. Let $g : \mathcal{G} \to \mathbb{N}$ be a Gödel numbering for labeled graphs of the form $G = \langle [n], E, <_{nat}\rangle$.

We define a graph polynomial using g:

$$\Gamma(G, X) = \sum_{H \simeq G} X^{g(H)}$$

Clearly this is a graph invariant. But it is *"obviously unnatural"*! Can we make precise what a *natural* graph polynomial should be?

1.8 Comparing graph invariants: getting started

In the literature we often find statements or questions of the form

- The Tutte polynomial is generalization of the chromatic polynomial.
- The Tutte polynomial does not determine the matching polynomial.
- Is there a natural most general graph polynomial?

We attempt to make this precise.

1.8.1 Induced graph invariants

Let $\mathcal{C} \subseteq \mathcal{G}$ be a class of graphs closed under isomorphism. Let F be a set of graph invariants taking values in in a ring $\mathcal{R}$, and let g be a further such graph invariant.

We say that F *induces* g *on* $\mathcal{C}$, or g *is a consequence of* F, if for any two graphs $G_1, G_2 \in \mathcal{C}$ such that $f(G_1) = f(G_2)$ for all $f \in F$ we also have $g(G_1) = g(G_2)$.

We denote by $Ind^{\mathcal{C}}_{\mathcal{R}}(F)$ the set of graph invariants in $\mathcal{R}$ induced by F on $\mathcal{C}$. We also write $F \models^{\mathcal{C}}_{\mathcal{R}} g$ for $g \in Ind^{\mathcal{C}}_{\mathcal{R}}(F)$.

How do we see whether $F \models^{\mathcal{C}}_{\mathcal{R}} g$?

1.8.2 Algebraically derived invariants

Let f, g be two graph invariants in $\mathcal{R}$. Then the following are derived invariants of $F = \{f, g\}$:

- $f + g, f - g, f \times g$
- The formal derivative f'.
- Let $\phi : \mathcal{R}^2 \to \mathcal{R}$ be a function. Then $\phi(f, g)$ is induced by F.

1.8.3 More examples of induced graph invariants

- Invariants induced by the characteristic polynomial
- Invariants induced by the acyclic (matching) polynomial
- Invariants induced by the chromatic polynomial
- The acyclic polynomial and the characteristic polynomial
- The acyclic polynomial and the chromatic polynomial
- The chromatic polynomial and the Tutte polynomial
- The Tutte polynomial and the matching polynomials

Invariants induced by the characteristic polynomial. The characteristic polynomial $char(G, X)$ induces

- The number of vertices $|V|$.
- The number of edges $|E|$.
- The number of triangles of G.

We also have $char(K_{1,4}, X) = char(C_4 \sqcup E_1, X)$ but $K_{1,4}$ has no 2-matchings, whereas C_4 does. Hence the $char(G, X)$ does not induce the number of connected components $k(G)$ nor $dm(G, X)$.

Invariants induced by the acyclic (matching) polynomial. The acyclic polynomial $dm(G,X)$ induces

- The number of vertices $|V|$.
- The number of edges $|E|$.
- The number of perfect matchings.
- the matching generating polynomial.

On the other side $dm(E_n,X)=1$ for all $n\in\mathbb{N}$, whereas $char(E_n,X)=X^n$. Hence the $dm(G,X)$ does not induce the characteristic polynomial $char(G,X)$.

Invariants induced by the chromatic polynomial. The following are induced by $\chi(G,X)=\sum_{i=1}^{n}(-1)^{n-i}a_iX^i$:

- The cardinality of $V(G)=n$ is the degree of $\chi(G,X)$.
- The cardinality of $E(G)=m=a_{n-1}$.
- The chromatic number $\chi(G)$ is the smallest integer a such that $\chi(G,a)>0$.
- The number of connected components $k(G)$ is the multiplicity of zeros $X=0$.
- The number of blocks $b(G)$ is the multiplicity of zeros $X=1$.
- The girth $g=g(G)$ is given by the fact that for $0\le i\le g-2$ we have $a_{n-i}=\binom{E(G)}{i}$.

The acyclic polynomial and the characteristic polynomial.

Theorem 1.19 ([13])**.** *$char(G,X)=dm(G,X)$ iff G is a forest.*

In other words, the acyclic (matching defect) polynomial and the characteristic polynomial coincide on the class $\mathcal{F}$ of forests. We have

$$char(G,X)\models^{\mathcal{F}} dm(G,X) \text{ and } dm(G,X)\models^{\mathcal{F}} char(G,X).$$

and

$$char(G,X)\models^{\mathcal{F}} gm(G,X) \text{ and } gm(G,X)\models^{\mathcal{F}} char(G,X).$$

In general, neither graph invariant induces the other.

Adjoint polynomials.

Definition 1.20. The complement graph of the simple graph $G=(V,E)$ is the graph $\bar{G}=(V,V^2-D(V)-E)$.

For a graph polynomial $g=g(G,\bar{X})$ the *adjoint polynomial* $\hat{g}(G,\bar{X})$ of g is defined by $\hat{g}(G,\bar{X})=g(\bar{G},\bar{X})$.

WARNING: In the literature on the chromatic polynomial the definition of adjoint polynomials differs!

The acyclic polynomial and the chromatic polynomial.

Theorem 1.21 (E.J. Farrell and E.G. Whitehead Jr. 1992)**.** *For $\mathcal{C} = \mathcal{TF}$, the triangle free graphs, we have*

$$\hat{\chi}(G, X) \models^{\mathcal{TF}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{TF}} \hat{\chi}(G, X).$$

i.e., the acyclic (matching defect) polynomial and the adjoint chromatic polynomial mutually induce each other.

Note that $\chi(P_4) = \chi(K_{1,3})$, $P_4 \simeq \bar{P}_4$, but $m(P_4) \neq m(K_{1,3})$. On the other hand, $m(E_n) = 1$ for each $n \in \mathbb{N}$, and $\chi(E_n) = X^n$. Hence, in general, neither one induces the other.

The chromatic polynomial and Tutte polynomial.

1. The chromatic polynomial $\chi(G, X)$ is not induced by the Tutte polynomial $T(G, X, Y)$.
2. On connected graphs $\mathcal{C}$ we have $T(G, X, Y) \models^{\mathcal{C}} \chi(G, X)$
3. Tutte polynomial $T(G, X, Y)$ is not induced by the the chromatic polynomial $\chi(G, X)$.

Proof. (i) Let E_n be the graph with n vertices and no edges. We have $T(E_n, X, Y) = 1$ but $\chi(E_n, X) = X^n$.

(ii) (After W.T. Tutte, 1954) $\chi(G, X) = (-1)^{|V|-k(G)} X^{k(G)} T(G, 1-X, 0)$.

(iii) (After M. Noy, 2003) Let W_n be the wheel with n spokes. It is known that $T(G, X, Y) = T(W_n, X, Y)$ implies that $G \simeq W_n$ for all n. But there is a $G \not\simeq W_5$ with $\chi(G, X, Y) = \chi(W_5, X, Y)$. □

The Tutte polynomial and the matching polynomials

- The matching polynomial is not induced by the Tutte polynomial, even on connected planar graphs.
- The Tutte polynomial is not induced by the matching polynomial, even on connected planar graphs.

Proof. (i) For trees with n vertices t_n we have $T(t_n, X, Y) = X^{n-1}$. But it is easy to see that $K_{1,n-1}$ and P_n are both trees with n vertices and their matching polynomials differ, as $K_{1,n-1}$ has no 2-matching but P_n has for $n \geq 3$.

(ii) On the other hand $C_3 \sqcup_e C_5$ and $C_4 \sqcup_e C_4$ have the same matching polynomials (check by hand) but have different Tutte polynomials, as the Tutte polynomials counts cliques of given size. □

What do we learn? What do we ask?

- Polynomial graph invariants are still a mystery.
- Can we analyze the consequence relation for polynomial invariants?
- Can we identify "good invariants"?
- What are appropriate complexity classes for graph invariants?

1.9 Comparing graph parameters and graph polynomials: towards a general theory

This section has been jointly prepared with E.V. Ravve.

1.9.1 Graph parameters and graph polynomials

Let $\mathcal{R}$ be a (possibly ordered) ring or a field. For a set of indeterminates $\bar{X}$ we denote by $\mathcal{R}[\bar{X}]$ the polynomial ring over $\mathcal{R}$.

A *graph parameter* p is a function from the class of all finite graphs to $\mathcal{R}$ which is invariant under graph isomorphism. A *graph polynomial* p is a function from the class of all finite graphs to $\mathcal{R}[\bar{X}]$ which is invariant under graph isomorphism.

Remark: In most situations in the literature $\mathcal{R}$ is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The choice of the underlying ring or field may depend on the way we want to represent the graph parameter or graph polynomial. For the graph parameter $d_{max}(G)$, the maximal degree of its vertices, $\mathbb{Z}$ suffices, but for $d_{average}(G)$, the average degree of its vertices, $\mathbb{Q}$ is needed.

1.9.2 Equivalence of graph polynomials

Let $\mathcal{C}$ be a graph property. Let $P(G, \bar{X})$ and let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 1.22. We say that Q *determines (induces)* P *over* $\mathcal{C}$, or Q *is at least as distinctive than* P *over* $\mathcal{C}$, and write $P \preceq^{\mathcal{C}}_{d.p.} Q$ if for all graphs G_1 and G_2 in $\mathcal{C}$,

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = P(G_2).$$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.

P and Q are *d.p.-equivalent* over $\mathcal{C}$, and write $P \sim^{\mathcal{C}}_{d.p.} Q$, iff $P \preceq^{\mathcal{C}}_{d.p.} Q$ and $Q \preceq^{\mathcal{C}}_{d.p.} P$.

Examples of $P \preceq^{\mathcal{C}}_{d.p.} Q$.

1. [8, 3.2.1] The chromatic polynomial $\chi(G, X)$ determines the graph parameters $|V(G)|$, $|E(G)|$, $\chi(G)$, $k(G)$, $b(G)$, $g(G)$, etc.
2. d_{max} and $d_{average}$ are d.p.-incomparable.
3. The Tutte polynomial $T(G, X, Y)$ determines $\chi(G, X)$ on connected graphs, but not on all graphs.
4. Assume $P(G; X), Q(G; X), U(G, X)$ are three polynomials and $P(G, X) = U(G, X) \cdot Q(G, X)$. Let $\mathcal{C}_U$ be a class of graphs such that for all $G_1, G_2 \in \mathcal{C}_U$, we have $U(G_1, X) = U(G_2 : X)$. Then $P \preceq^{\mathcal{C}_U}_{d.p.} Q$.
5. Let $\mathcal{F}$ be the class of forests. For the characteristic polynomial $char(G, \lambda)$ and the matching polynomial $dm(G, \lambda)$ and we have

$$char \sim^{\mathcal{F}}_{d.p.} dm \; .$$

1.9.3 P-unique and P-equivalent graphs

Definition 1.23. Let $P = P(G; \bar{X})$ a graph polynomial and $\mathcal{C}$ a class of graphs.

1. Two graphs G_1 and G_2 are *P-equivalent for $\mathcal{C}$* if $P(G_1; \bar{X}) = P(G_2; \bar{X})$.
2. A graph $G \in \mathcal{C}$ is *P-unique for $\mathcal{C}$* if for any other graph $G_1 \in \mathcal{C}$ with $P(G; \bar{X}) = P(G_1; \bar{X})$ the graph G_1 is isomorphic to G.
3. *P is complete for $\mathcal{C}$* if every graph $G \in \mathcal{C}$ is $P(G; \bar{X})$-unique for $\mathcal{C}$.

If $\mathcal{C}$ consists of all graphs then we omit mention of $\mathcal{C}$.

Proposition 1.24. *Let P and Q be graph polynomials such that $P \preceq^{\mathcal{C}}_{d.p.} Q$.*

1. *If G_1 and G_2 are Q-equivalent for $\mathcal{C}$ then they are also P-equivalent for $\mathcal{C}$.*
2. *If G is P-unique for $\mathcal{C}$ then G is Q-unique for $\mathcal{C}$.*
3. *If P is complete for $\mathcal{C}$ then Q is complete for $\mathcal{C}$.*

1.9.4 More examples of induced graph invariants

- Adjoint polynomials
- χ-equivalent graphs [8, Chapter 5]
- The two matching polynomials
- T-unique graphs
- Almost complete graph invariants

Adjoint polynomials. Recall Definition 1.20 of the adjoint of a graph polynomial.

Exercise: $P \preceq^{\mathcal{C}}_{d.p.} \hat{P}$ iff $\hat{P} \preceq^{\mathcal{C}}_{d.p.} P$

For the Tutte polynomial $T(G, X, Y)$ and $\bar{E_n} = K_n$ we have

1. $T(E_m) = T(E_n) = 1$ for all $n \in \mathbb{N}$.
2. $T(K_m) \neq T(K_n)$ for $m \neq n$.
3. Hence the Tutte polynomial and its adjoint are not d.p.-comparable.

χ-equivalent graphs. From [8, Chapter 5].

1. The graphs E_n, K_n and $K_{n,n}$ are χ-unique for $n \geq 1$.
2. The graphs C_n are χ-unique for $n \geq 3$, $C_i = K_i$ for $i \leq 2$.
3. Any two trees on n vertices are χ-equivalent.

In [8, Chapter 5] many pairs of χ-equivalent graphs are constructed using a method due to R.C. Read (1987) and G.L. Chia (1988).

Research project: Study P-equivalence for the various generalized colorings of [14].

***char*-equivalent graphs.** From [21]. Let $char(G, x) = \det(x \cdot \mathbf{1} - A_G)$ be the characteristic polynomial of G with adjacency matrix A_G.

1. The graphs $K_{n,n}$ are *char*-unique.
2. The line graphs $L(K_n)$ are *char*-unique for $n \neq 8$. For $n = 8$ there are three exceptions.
3. The line graphs $L(K_{n,n})$ are *char*-unique for $n \neq 4$. For $n = 4$ there is one exception.

The two matching polynomials. Recall the relation (4) between (2) and (3).

Graphs equivalent for matching polynomials. From [21].

- For every graph G we have $gm(G, x) = gm(G \sqcup E_n, x)$
  but $dm(G, x) \neq dm(G \sqcup E_n, x)$.
  $dm(P_2, x) = x^2 - 1$ and $dm(P_2 \sqcup E_k, x) = x^3 - x$,
  but $gm(P_2, x) = x^2 - 1$ and $gm(P_2 \sqcup E_k, x) = x^2 - 1$
- $|V(G)| \preceq_{d.p.} dm$, and therefore $gm \preceq_{d.p.} dm$.
  In other words gm is strictly less expressive than dm.
- $gm \sim_{d.p.} dm$ on graphs with a fixed number of vertices.
- The graphs $K_{n,n}$ are dm-unique.
  Are they also gm-unique?

Research project: Study dm-equivalence and gm-equivalence of graphs further.

T-unique graphs. From [20].

For a graph $G = (V, E)$ and $A \subseteq E$ we denote by $G[A] = (V, A)$ the spanning subgraph generated by A. We set $r(A) = |V| - k(G[A])$ and $n(A) = |A| - r(A)$. The Tutte polynomial (5) can be written as

$$T(G; X, Y) = \sum_{A \subseteq E} (X - 1)^{r(E) - r(A)} (Y - 1)^{n(A)}.$$

1. Recall that $\chi \preceq_{d.p.} T$ on connected graphs. Hence the graphs $K_{n,n}$ are T-unique.
2. The wheels W_n are T-unique for all $n \in \mathbb{N}$. Wheels are χ-unique for W_{2n}, W_5 and W_7 are not. In general it is not known (?) whether W_{2n+1} is χ-unique.
3. The ladders L_n are T-unique for all $n \geq 3$. They are only known to be χ-unique for small values of n.

Bollobás–Pebody–Riordan Conjecture [3]: Almost all graphs are T-unique and even χ-unique.

Let us make it more precise. Let TU (χU) be the graph property:

$$G \in TU \ (G \in \chi U) \text{ iff } G \text{ is } T\text{-unique } (\chi\text{-unique}),$$

and $TU(n)$ ($\chi U(n)$) be the density function of TU (χU).

The conjecture for the Tutte polynomial now is

$$\lim_{n \to \infty} \frac{TU(n)}{2^{\binom{n}{2}}} = 1$$

Similarly for $\chi U(n)$.

Is TU (χU) definable in some logic with a zero-one law?

Almost complete graph invariants. A graph polynomial P is *almost complete*, if almost all graphs are P-unique.

Research problems:

- Study the definability of the graph property G *is* P *unique* for various graph polynomials P.
- Find natural graph polynomials which are almost complete.
- In particular, is the *signed Tutte polynomial* T_{signed} almost complete for signed graphs?
  A positive answer would be interesting for knot theorists: T_{signed} is intimately related to the Jones polynomial of knot theory.

1.10 Comparison of graph polynomials by coefficients

1.10.1 Coefficients of graph polynomials: the univariate case

We denote by $\mathbb{Z}^{<\omega}$ the set of finite sequences of elements of $\mathbb{Z}$. Let $P(G,X) \in \mathbb{Z}[X]$ and $P(G,X) = \sum_{i=0}^{|V(G)|} a_i(G) \cdot X^i$ with $a(G)_{|V(G)|} \neq 0$.

We denote by $cP(G,X)$ the finite sequence $(a_i(G))_{i \leq |V(G)|} \in \mathbb{Z}^{<\omega}$. The sequence $cP(G,X)$ consists of the (standard) coefficients of $P(G,X)$. The function $c : \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$ is one-to-one and onto.

Instead of looking at graph polynomials $P : \mathit{Graphs} \xrightarrow{P} \mathbb{Z}[X]$, we can look at the function $cP : \mathit{Graphs} \longrightarrow \mathbb{Z}^{<\omega}$ defined by

$$cP : \mathit{Graphs} \xrightarrow{P} \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$$

Lemma 1.25. *For all graphs G_1, G_2, we have that $P(G_1) = P(G_2)$ iff $cP(G_1) = cP(G_2)$.*

Our definition of cP uses the *power form of* P. We could have used also *factorial form* or *binomial form* of P.

- cP denotes the coefficients of P in power form.
- c_1P denotes the coefficients of P in factorial form.
- c_2P denotes the coefficients of P in binomial form.

We note that there are simple algorithms to pass from one representation to another.

1.10.2 Equivalence of graph polynomials: coefficients

Let $\mathcal{C}$ be a graph property. Let $P(G,\bar{X})$ and let $Q(G,\bar{Y})$ be two graph polynomials.

Definition 1.26. We say that Q *determines coefficientwise* P *over* $\mathcal{C}$ and write $P \preceq^{\mathcal{C}}_{coeff} Q$ if there is a function $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$

$$F(cQ(G)) = cP(G)$$

P and Q are *coefficient-equivalent* over $\mathcal{C}$, and write $P \sim^{\mathcal{C}}_{coeff} Q$, iff $P \preceq^{\mathcal{C}}_{coeff} Q$ and $Q \preceq^{\mathcal{C}}_{coeff} P$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.
- Our definition is invariant under the choice of representations cP, c_1P or c_2P.

An example: F can be arbitrarily complex. Let $P(G,\lambda) = \sum_i a_i(G)\lambda^i$. Let $P_{exp}(G,\lambda) = \sum_i 2^{a_i(G)}\lambda^i$, and for $g : \mathbb{N} \to \mathbb{N}$ one-to-one and onto let $P_g(G,\lambda) = \sum_i a_i(G)\lambda^{g(i)}$.

Clearly,

$$P \sim_{coeff} P_g \sim_{coeff} P_{exp}$$

- If g is not computable, then F showing that $P \sim_{coeff} P_g$ cannot be computable in the *Turing model* of computation.
- Furthermore, F showing that $P \sim_{coeff} P_{exp}$ cannot be computable in the *Blum-Shub-Smale model* of computation.

Theorem 1.27. *$P \preceq^{\mathcal{C}}_{coeff} Q$ iff $P \preceq^{\mathcal{C}}_{d.p.} Q$*

Proof. $P \preceq^{\mathcal{C}}_{coeff} Q$ **implies** $P \preceq^{\mathcal{C}}_{d.p.} Q$**.**

Assume there is a function $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$.

Now let $G_1, G_2 \in \mathcal{C}$ such that $Q(G_1) = Q(G_2)$. By Lemma 1.25 we have $cQ(G_1) = cQ(G_2)$. Hence $F(cQ(G_1)) = F(cQ(G_2))$.

Since for all $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$, we get $cP(G_1) = cP(G_2)$ and, using Lemma 1.25 again, we have $P(G_1) = P(G_2)$.

$P \preceq^{\mathcal{C}}_{d.p} Q$ **implies** $P \preceq^{\mathcal{C}}_{coeff} Q$**.**

We use the well-ordering principle, which is equivalent to the axiom of choice.

Let $\{F_\alpha : \alpha < \beta\}$ be a well-ordering of all functions $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$. For $G \in \mathcal{C}$, let $\gamma(G) < \beta$ be the smallest ordinal such that $F_{\gamma(G)}(cQ(G)) = cP(G)$.

Now given $P(G,X) \preceq_{d.p.} Q(G,X)$, we define a function $F_{P,Q} : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ as follows:

$$F_{P,Q}(cQ(G)) = \begin{cases} F_{\gamma(G)}(cQ(G)) & \text{if } G \in \mathcal{C} \\ 0 & \text{else} \end{cases}$$

Using Lemma 1.25 and $P(G,X) \preceq_{d.p.} Q(G,X)$, this indeed defines a function. Finally, as $F_{\gamma(G)}(cQ(G)) = F_{\gamma(G)}(cP(G))$, we get $F_{P,Q}(cQ(G)) = cP(G)$. $\square$

A proof without well-ordering (suggested by Ofer David). Let S be a set of finite graphs and $s \in \mathbb{Z}^{<\omega}$. For a graph polynomial P we define: $P[S] = \{s \in \mathbb{Z}^{<\omega} : cP(G) = s \text{ for some } G \in S\}$ and $P^{-1}(s) = \{G : cP(G) = s\}$.

Now assume $P(G,X) \preceq_{d.p.} Q(G,X)$. If $Q^{-1}(s) \neq \emptyset$, then for every $G_1, G_2 \in Q^{-1}(s)$ we have $cQ(G_1) = cQ(G_2)$, and therefore $cP(G_1) = cP(G_2)$. Hence $P[Q^{-1}(s)] = \{t_s\}$ for some $t_s \in \mathbb{Z}^{<\omega}$.

Now we define

$$F_{P,Q}(s) = \begin{cases} t_s & Q^{-1}(s) \neq \emptyset \\ s & \text{else} \end{cases}$$

and the argument is completed $\square$

Example I: the two matching polynomials. Recall we have $dm(G;x) = x^n gm(G;(-x)^{-2})$ for (2) and (3) where $n = |V(G)|$.

- The degree of dm is n
- If $m_r(G) \neq 0$ the $n - 2r > 0$.
- Hence
$$\frac{dm(G;x)}{X^n}$$
is a polynomial, and we can compute the coefficients of gm from the coefficents of dm.
- We cannot compute the coefficients of dm from gm without knowing the value of $|V(G)| = n$.

Example II: the Tutte polynomial and the chromatic polynomial. The Tutte polynomial and the chromatic polynomial are related by the formula

$$\chi(G,X) = (-1)^{r(G)} \cdot X^{k(G)} \cdot T(G;1-X,0)$$

- To compute the coefficients of $\chi(G;X)$ from $T(G;X,Y)$ we have to know the parity of $r(G)$ and the number of connected components of G.
- For connected graphs $k(G) = 1$ and $r(G) = |V| - 1$.

1.10.3 Introducing auxiliary parameters $\mathcal{S}$

Let $\mathcal{S} = \{S_1(G), \ldots, S_t(G)\}$ be graph parameters (polynomials), and $\mathcal{C}$ a graph property. Let $P(G,\bar{X})$ and $Q(G,\bar{Y})$ be two graph polynomials.

Definition 1.28. We say that Q *determines* P *relative to* $\mathcal{S}$ *over* $\mathcal{C}$, or Q *is at least as distinctive as* P *relative to* $\mathcal{S}$ *over* $\mathcal{C}$, and write

$$P \preceq^{\mathcal{S},\mathcal{C}}_{r.d.p.} Q$$

if for all graphs $G_1, G_2 \in \mathcal{C}$ with $S_i(G_1) = S_i(G_2) : i \leq t$ we have

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = Q(P_2).$$

Definition 1.29. We say that Q *determines* P *coefficientwise relative to* $\mathcal{S}$ *over* (C), and write

$$P \preceq^{\mathcal{S},(C)}_{relcoeff} Q,$$

if there is a function $F : (\mathbb{Z}^{<\omega})^{t+1} \rightarrow \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{P}$

$$F(cS_1(G), \ldots, cS_t(G), cQ(G)) = cP(G).$$

The equivalence relations $P \sim^{\mathcal{S},(C)}_{r.d.p.} Q$ and $P \sim^{\mathcal{S},(C)}_{relcoeff} Q$, are defined as usual.

Theorem 1.30. $P \preceq^{\mathcal{S}}_{relcoeff} Q$ *iff* $P \preceq^{\mathcal{S}}_{r.d.p.} Q$.

The proof is left as an exercise!

1.11 Semantic properties of graph polynomials

1.11.1 Similar graphs and similarity functions

Two graphs G_1, G_2 are *similar* if they have the same number of vertices, edges and connected components, i.e.,

- $|V(G_1)| = n(G_1) = n(G_2) = |V(G_2)|$,
- $|E(G_1)| = m(G_1) = m(G_2) = |E(G_2)|$, and
- $k(G_1) = k(G_2)$.
- $\mathcal{S} = \{|V(G)|, |E(G)|, k(G)\}$

A graph parameter or graph polynomial is a *similarity function* if it is invariant under similarity.

1. The nullity $\nu(G) = m(G) - n(G) + k(G)$ and the rank $\rho(G) = n(G) - k(G)$ of a graph G are similarity polynomials with integer coefficients.
2. Similarity polynomials can be formed inductively starting with similarity functions $f(G)$ not involving indeterminates, and monomials of the form $X^{g(G)}$ where X is an indeterminate and $g(G)$ is a similarity function not involving indeterminates. One then closes under pointwise addition, subtraction, multiplication and substitution of indeterminates X by similarity polynomials.

1.11.2 Comparing graph polynomials up to graph similarity

In the literature graph polynomials are mostly compared up to graph similarity:

- We note that the various matching polynomials are *not d.p.-equivalent*. The number of vertices of a graph G is not induced by all its variations, but is induced by some of them.
- However, if restricted to similar graphs, all the matching polynomials have the same distinctive power.
- Similarily, the Tutte polynomial does not induce the chromatic polynomial. They behave differently on empty graphs. However, on similar graphs, the Tutte polynomial determines the chromatic polynomial.

This leads to the following definitions.

Two graph polynomials are usually compared via their distinctive power.

A graph polynomial $Q(G, X)$ is *less s-distinctive than* $P(G, Y)$, $Q \preceq_s P$, if for every two similar graphs G_1 and G_2

$$P(G_1, X) = P(G_2, X) \text{ implies } Q(G_1, Y) = Q(G_2, Y).$$

We also say the $P(G; X)$ *s-determines* $Q(G; X)$ if $Q \preceq_s P$.

Two graph polynomials $P(G, X)$ and $Q(G, Y)$ are *s-equivalent in distinctive power (s.d.p-equivalent)* if for every two similar graphs G_1 and G_2

$$P(G_1, X) = P(G_2, X) \text{ iff } Q(G_1, Y) = Q(G_2, Y).$$

The same definition also works for graph parameters and multivariate graph polynomials.

Let $\mathcal{R}$ be the ring of coefficients of our graph polynomials, and let $\mathcal{R}^{<\omega}$ denotes the set of finite sequences of $\mathcal{R}$. We denote by $cP(G) \in \mathcal{R}^{<\omega}$ the sequence of coefficients of $P(G, X)$.

Proposition 1.31. *Two graph polynomials $P(G, X_1, \ldots X_r)$ and $Q(G, Y_1, \ldots, Y_s)$ are* s-equivalent in distinctive power (s.d.p-equivalent) over $\mathcal{S}$ *($P \sim_{s.d.p.} Q$) iff there are two functions $F_1, F_2 : \mathcal{R}^{<\omega} \to \mathcal{R}^{<\omega}$ such that for every graph G*

$$F_1(n(G), m(G), k(G), cP(G)) = cQ(G) \text{ and}$$
$$F_2(n(G), m(G), k(G), cQ(G)) = cP(G)$$

Proposition 1.31 shows that our definition of equivalence of graph polynomials is mathematically equivalent to the definition proposed by C. Merino and S. Noble in 2009.

Computability. The functions F_1, F_2 in Proposition 1.31 need not be computable in any sense, even if the coefficients of $P(G)$ and $Q(G)$ are integers.

A graph polynomial $P(G; X)$ with coefficients in a ring $\mathcal{R}$ is *computable* (in a suitable model of computation for $\mathcal{R}$) if

1. the function $cP : \mathcal{G} \to \mathcal{R}^{<\omega}$ computing the coefficients of $P(G; X)$ is computable, and
2. the decision problem *given $s \in \mathcal{R}^{<\omega}$ is there a graph with $cP(G) = s$* is decidable.

Theorem 1.32. *Let $P(G; X)$ and $Q(G; X)$ be two computable graph polynomials which are s.d.p.-equivalent.Then there are F_1, F_2 as in Proposition 1.31 which are computable.*

In this case we say that $P(G; X)$ and $Q(G; X)$ are *computably s.d.p.-equivalent.*

1.11.3 Prefactor and subtsitution equivalence

We say that $P(G; \bar{X})$ is *prefactor reducible to* $Q(G; \bar{X})$ and we write

$$P(G; \bar{Y}) \preceq_{prefactor} Q(G; \bar{X})$$

if there are *similarity functions* $f(G; \bar{X}), g_1(G; \bar{X}), \ldots, g_r(G; \bar{X})$ such that

$$P(G; \bar{Y}) = f(G; \bar{X}) \cdot Q(G; g_1(G; \bar{Y}), \ldots, g(G; \bar{Y})).$$

We say that $P(G; \bar{X})$ is *substitution reducible to* $Q(G; \bar{X})$, and we write

$$P(G; \bar{Y}) \preceq_{subst} Q(G; \bar{X})$$

if $P(G; \bar{Y}) \preceq_{prefactor} Q(G; \bar{X})$ and, additionally, $f(G; \bar{X}) = 1$ for all values of $\bar{X}$.

$P(G; \bar{X})$ and $Q(G; \bar{X})$ are prefactor (substitution) equivalent if the relationship holds in both directions.

It follows that if the computable graph polynomials $P(G; \bar{X})$ and $Q(G; \bar{X})$ are prefactor (substitution) equivalent then they are computably s.d.p.-equivalent.

1.11.4 Semantic properties of graph parameters

A *semantic property* (*s-semantic property*) is a class of graph parameters (polynomials) closed under *d.p.-equivalence* or (*s.d.p.-equivalence*).

Let $p(G)$ be a graph parameter with values in $\mathbb{N}$, and $P(G; X)$ be a graph polynomial.

- The degree of $P(G; X)$ equals $p(G)$ is not a semantic property of $P(G; X)$. Using Proposition 1.31 we see that $P(G; X)$ and $P(G; X^2)$ are d.p.-equivalent, but they have different degrees.
- $P(G; X)$ determines $p(G)$ is a semantic property of $P(G; X)$. The degree of $P(G; X)$ equals $p(G)$ is an accidental result of the particular presentation of $P(G; X)$.
- The number of triangles of G is determined by the characteristic polynomial, but that it is twice the absolute value of the third coefficient again is a result of its particular presentation.

1.11.5 Semantic vs syntactic properties of graph polynomials

Semantically meaningless properties:

1. $P(G, X)$ is monic for each graph G, i.e., the leading coefficient of $P(G; X)$ equals 1.
   Multiplying each coefficient by a fixed polynomial gives an equivalent graph polynomial.
2. The leading coefficient of $P(G, X)$ equals the number of vertices of G. However, proving that two graphs G_1, G_2 with $P(G_1, X) = P(G_2, X)$ have the same number of vertices is semantically meaningful.
3. The graph polynomials $P(G; X)$ and $Q(G; X)$ coincide on a class $\mathcal{C}$ of graphs, i.e. for all $G \in \mathcal{C}$ we have $P(G; X) = Q(G; X)$. (Theorem 1.19 provides an example with P the characteristic polynomial, Q the acyclic (defect matching) polynomial, and $\mathcal{C}$ the class of forests.)
   The semantic content of this situation says that if we restrict our graphs to $\mathcal{C}$, then $P(G; X)$ and $Q(G; X)$ have the same distinguishing power.
   The equality of $P(G; X)$ and $Q(G; X)$ is a syntactic coincidence or reflects a clever choice in the definitions $P(G; X)$ and $Q(G; X)$.

Such clever choices can be often achieved.

Let $\mathcal{C}$ be a class of finite graphs closed under graph isomorphism.

Proposition 1.33. *Suppose that $P(G; X)$ and $Q(G; X)$ are graph polynomials that have the same distinguishing power on the class of graphs $\mathcal{C}$, i.e. $P \sim^{\mathcal{C}}_{d.p.} Q$. Then there is a graph polynomial P' with the same distinguishing power as P on all graphs, i.e. $P' \sim_{d.p.} P$, such that $P'(G; X) = Q(G; X)$ on $\mathcal{C}$.*

If, additionally, $\mathcal{C}, P(G; X)$ and $Q(G; X)$ are computable, then $P'(G; X)$ can be chosen to be computable too.

Proposition 1.33 is simply satisfied by choosing $P' = Q$ on $\mathcal{C}$ and $P' = P$ outside $\mathcal{C}$ (and if P, Q and $\mathcal{C}$ are computable then P' is computable too).

Proposition 1.33 also holds when we replace *computable* by *definable in* SOL.

2 Why is the chromatic polynomial a polynomial?

Taming the class of graph polynomials. Definability of graph properties and numeric graph parameters

- Variations of the chromatic polynomial
- Why are there many chromatic polynomials?
- The classical graph polynomials
- The matching polynomials: a case study
- Second-order logic (SOL)
- Graph properties
- Logic and complexity
- HEX and variations
- The role of order
- Definability of numeric graph invariants

2.1 The chromatic polynomial and its variations

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$. A λ-*vertex-coloring* is a map $c : V(G) \to [\lambda]$ such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$. We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings. Recall Section 1.2.

Theorem 2.1 (G. Birkhoff, 1912)**.** $\chi(G, \lambda)$ *is a polynomial in* $\mathbb{Z}[\lambda]$.

Interpretation of $\chi(G, \lambda)$ for $\lambda \notin \mathbb{N}$. What's the point in considering $\lambda \notin \mathbb{N}$?

- (Stanley, 1973). For simple graphs G, $|\chi(G, -1)|$ counts the number of acyclic orientations of G.
- (Stanley, 1973). There are also combinatorial interpretations of $\chi(G, -m)$ for each $m \in \mathbb{N}$, which are more complicated to state.
- Open:What about $\chi(G, \lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

The Four Color Conjecture. Birkhoff wanted to prove the Four Color Conjecture using techniques from real and complex analysis.

Conjecture: (Birkhoff and Lewis) If G is planar then $\chi(G, \lambda) \neq 0$ for $\lambda \in [4, +\infty) \subseteq \mathbb{R}$.

This was not very successful. However, for real roots of χ we know:

- (Jackson, 1993). For simple graphs G we have $\chi(G, \lambda) \neq 0$ for $\lambda \in (-\infty, 0)$, $\lambda \in (0, 1)$ and $\lambda \in (1, \frac{32}{27})$.
- (Birkhoff and Lewis, 1946). For planar graphs G we have $\chi(G, \lambda) \neq 0$ for $\lambda \in [5, +\infty)$.
- Still open: Are there planar graphs G such that $\chi(G, \lambda) = 0$ for some $\lambda \in (4, 5)$?
- (Thomassen, 1997 and Sokal, 2004). The real roots of all chromatic polynomials are dense in $(\frac{32}{27}, k]$ for graphs of tree-width at most k. The complex roots are dense in $\mathbb{C}$.

2.1.1 Variations on coloring

We can count other coloring functions.

1. Proper λ-edge-colorings:
   $f_E : E(G) \to [\lambda]$ such that if $(e, f) \in E(G)$ have a common vertex then $f_E(e) \neq f_E(f)$.
   $\chi_e(G, \lambda)$ denotes the number of λ- edge-colorings
2. Total colorings:
   $f_V : V \to [\lambda_V]$, $f_E : E \to [\lambda_E]$ and $f = f_V \cup f_E$, with f_V a proper vertex coloring and f_E a proper edge coloring.
3. Connected components:
   $f_V : V \to [\lambda_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.
4. Hypergraph colorings. See [23].

Fact: The corresponding counting functions to 1-4 are polynomials in λ.

Let $f : V(G) \to [\lambda]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, \lambda)$ denotes the number of such colorings with at most λ colors.

* **convex:** Every monochromatic set induces a connected graph.

* **injective:** f is injective on the neighborhood of every vertex.

\- **complete:** f is a proper coloring such that every pair of colors occurs along some edge.

* **harmonious:** f is a proper coloring such that every pair of colors occurs at most once along some edge.

\- **equitable:** All color classes have (almost) the same size.

* **equitable, modified:** All non-empty color classes have the same size.

Fact: For (*), $\chi_\Phi(G, \lambda)$ is a polynomial in λ, for (-), it is not.

* **path-rainbow:** Let $f : E \to [\lambda]$ be an edge-coloring. f is *path-rainbow* if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

Fact: $\chi_{rainbow}(G, \lambda)$, the number of path-rainbow colorings of G with λ colors, is a polynomial in λ. Rainbow colorings of various kinds arise in computational biology.

* **-monochromatic components:** Let $f : V \to [\lambda]$ be a vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with λ colors if all the connected components of a monochromatic set have size at most t.

Fact: For fixed $t \geq 1$ the function $\chi_{mcc_t}(G, \lambda)$ counting the number of mcc_t-colorings of G with λ colors is a polynomial in λ but not in t. mcc_t-colorings were first studied in [1].

Let $\mathcal{P}$ be any graph property and let $n \in \mathbb{N}$. We can define coloring functions $f : V \to [\lambda]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the *convex colorings*.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs we get the *acyclic colorings*, introduced in [12].

Theorem 2.2. *Let $\chi_{\mathcal{P},n}(G, \lambda)$ be the number of colorings of G with λ colors such that the union of any n color classes induces a graph in $\mathcal{P}$.*

Then $\chi_{\mathcal{P},n}(G, \lambda)$ is a polynomial in λ.

Variations on colorings: coloring relations Let $G = (V, E)$. Here we look at an example where the coloring is a relation $R \subseteq V \times [k]$ rather than a function $f : V \to [k]$. We denote by C_v the set $\{c \in [k] : (v, c) \in R\}$.

Let $a, b \in \mathbb{N}$. An *(a, b)-coloring relation with k colors* is a relation $R \subseteq V \times [k]$ such that

- For each $v \in V$ there are at most a-many colors $c \in [k]$ such that $(v, c) \in R$.
- If $(u, v) \in E$ then $C_u \neq C_v$ and there are at most b-many distinct elements $c_1, \ldots, c_b$ in $C_u \cap C_v$.

Exercise:

- Compute the number of (a, b)-coloring relations of the complete graphs K_n for various $a, b, k \in \mathbb{N}$.
- Is the number (a, b)-coloring relations with k colors of a graph G a polynomial in a, b or k?
- Look at the corresponding definitions with "at most" replaced by "at least" or "exactly".

Variations on colorings: two kinds of colors. Let $G = (V, E)$. Here we look at two disjoint color sets $A = [k_1]$ and $B = [k_1 + k_2] - [k_1]$. The colors in A are called *proper* colors. Our coloring is a function $f : V \to [k_1 + k_2] = [k]$ such that

- If $(u, v) \in E$ and $f(u) \in A$ and $f(v) \in A$ then $f(u) \neq f(v)$.
- We count the number of colorings with $k = k_1 + k_2$ colors such that k_1 colors are in A, i.e., proper.

Theorem 2.3 (K. Dohmen, A. Pönitz and P. Tittman, 2003)**.** *The number of colorings with $k = k_1 + k_2$ colors with k_1 proper colors is a polynomial $P(G, k_1, k)$ in k_1 and k.*

2.2 Coloring properties. Why are there many chromatic polynomials?

Our framework is as follows:

- Let $\mathfrak{M}$ be a finite relational τ-structure with universe M.
- Let $k \in \mathbb{N}$ and $[k] = \{0, \ldots, k - 1\}$.
- Let f be an r-ary function $f : M^r \to [k]$.

- We shall look at families $\mathcal{P}$ consisting of triples of the form $(\mathfrak{M}, f, [k])$.

A class of such triples $\mathcal{P}$ is a *coloring property* if the following properties are satsified:

Extension Property: Let $n \leq k_i, i = 1, 2$, and let

$(\mathfrak{M}, f, [k_1])$ and $(\mathfrak{M}, f, [k_2])$ be two colorings of $\mathfrak{M}$,

using only colors in $[n]$, i.e., the range of f is contained in $[n]$.

Then $(\mathfrak{M}, f, [k_1]) \in \mathcal{P}$ iff $(\mathfrak{M}, f, [k_2]) \in \mathcal{P}$.

Isomorphism Property: $\mathcal{P}$ is closed under isomorphisms of colorings.

The isomorphism property implies the permutation property:

Permutation Property: Let $f : M^r \to [k]$ be a fixed coloring.

For a permutation π of $[k]$, we define the coloring f_π by $f_\pi(\bar{a}) = \pi(f(\bar{a}))$.

Then $(\mathfrak{M}_k, f, [k]) \in \mathcal{P}$ iff $(\mathfrak{M}_k, f_\pi, [k]) \in \mathcal{P}$.

If instead of coloring functions f we allow coloring relations $R \subseteq M^r \times [k]$ we need some additional properties:

1. A coloring property $\mathcal{P}$ of triples $(\mathfrak{M}_k, R, [k]) \in \mathcal{P}$ is *bounded*, if for every $\mathfrak{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors

$$\{x \in [k] : \exists \bar{y} \in M^r \ R(\bar{y}, x)\}$$

has size at most N_M.

2. A coloring property is *range bounded* if there is a number $d \in \mathbb{N}$ such that for every $\mathfrak{M}$ and $\bar{y} \in M^r$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.

Clearly, if a coloring property is range bounded, it is also bounded.

We denote by $\chi_{\mathcal{P}}(\mathfrak{M}, k)$ the number of generalized $k - \mathcal{P}$-colorings R on $\mathfrak{M}$. The work presented in Sections 2.2.1 and 2.2.2 is based on [14].

2.2.1 Uniform definability in logical formalisms

Let ϕ be a sentence of some logic $\mathcal{L}$, such as first order logic FOL, second order logic SOL, monadic second order logic MSOL, or some fragment thereof.[2]

We shall be interested in cases where the coloring property $\mathcal{P}$ is definable in $\mathcal{L}$ by a formula $\phi(P) \in \mathcal{L}$. If $\phi(P)$ defines a (bounded) coloring property, we say that $\phi(P)$ is a *coloring formula.*

If $\mathcal{P}$ is $\mathcal{L}$-definable we call $\chi_{\mathcal{P}}(\mathfrak{M}, k)$ an $\mathcal{L}$*-chromatic counting function* and write

$$\chi_{\phi(P)}(\mathfrak{M}, k) = \chi_{\mathcal{P}}(\mathfrak{M}, k).$$

All the examples encountered so far are SOL-chromatic counting functions.

[2] *Editors' note*: See Section 3.2 for the definition of a *fragment.*

2.2.2 Generalized multi-colorings

To construct graph polynomials in several variables, we extend the definition of colorings to several color-sets, and we will also call them generalized chromatic polynomials.

We say an $(\alpha+2)$-tuple $(\mathfrak{M}, R, [k_1], \ldots, [k_\alpha])$ with

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

is a *generalized multi-coloring* of $\mathfrak{M}$ for colors $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$.

The extension and isomorphism property are adapted appropriately to deal also with unused color-sets. By abuse of notation, $m_i = 0$ is taken to mean the color-set k_i is not used in R.

Theorem 2.4. *For every $\mathfrak{M}$ the counting function $\chi_{\phi(R)}(\mathfrak{M}, k)$ is a polynomial in k of the form*

$$\sum_{j=0}^{d\cdot|M|^m} c_{\phi(R)}(\mathfrak{M}, j)\binom{k}{j}$$

where $c_{\phi(R)}(\mathfrak{M}, j)$ is the number of generalized $k-\phi$-colorings R with a fixed set of j colors.

Polynomials in $\mathbb{Z}[k]$ with monomials of the form $\binom{k}{j}$ are sometimes called Newton polynomials. In the light of this theorem we call $\chi_{\phi(R)}(\mathfrak{M}, k)$ a *generalized chromatic polynomial*.

Proof. We first observe that any generalized coloring R uses at most

$$N = d \cdot |M|^m$$

of the k colors. For any $j \leq N$, let $c_{\phi(R)}(\mathfrak{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalized vertex colorings and use all j of the colors.

Next we observe that any permutation of the set of colors used is also a coloring. Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\phi(R)}(\mathfrak{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\phi(R)}(\mathfrak{M}, k) = \sum_{j\leq N} c_{\phi(R)}(\mathfrak{M}, j)\binom{k}{j}$$

The right-hand side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k < j$ we have $\binom{k}{j} = 0$. $\square$

2.3 More on classical graph polynomials

Here again are some of the classical graph polynomials:

- The chromatic polynomial (G. Birkhoff, 1912)

- The Tutte polynomial and its colored versions
  (W.T. Tutte, 1954; B. Bollobas and O. Riordan, 1999);
- The characteristic polynomial
  (T.H. Wei, 1952; L.M. Lihtenbaum, 1956; L. Collatz and U. Sinogowitz, 1957)
- The various matching polynomials (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials (I. Gutman and F. Harary, 1983)
- The Farrel polynomials (E.J. Farrell, 1979)
- The cover polynomials for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The interlace polynomials (M. Las Vergnas, 1983; R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various knot polynomials (of signed graphs) (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc.)

As we said before, there are plenty of applications of classical graph polynomials in

- Graph theory proper and knot theory;
- Chemistry and biology;
- Statistical mechanics (Potts and Ising models);
- Social networks and finance mathematics;
- Quantum physics and quantum computing.

And what about the *many* other graph polynomials we have just learned to construct?

Let us briefly summarize the different ways in which graph polynomials can be compared:

- By **distinctive power**:
  $P(G;\bar{X}) \preceq_{d.p.} Q(G;\bar{X})$ if for any two graphs G_1, G_2 with $Q(G_1;\bar{X}) = Q(G_2;\bar{X})$ we also have $P(G_1;\bar{X}) = P(G_2;\bar{X})$
- By **coefficient computation**:
  $P(G;\bar{X}) \preceq_{coeff} Q(G;\bar{X})$ if there is a function F which computes for every G the coefficients of $P(G;\bar{X})$ from the coefficients of $Q(G;\bar{X})$.
- By **substitution instance**.
  $P(G;\bar{X}) \preceq_{subst} Q(G;\bar{X})$ if there is a substitution σ of the variables such that for every G $P(G;\bar{X}) = Q(G;\sigma(\bar{X}))$.

The first statement of the following proposition is a special case of Theorem 1.27.

Proposition 2.5. *$P(G;\bar{X}) \preceq_{d.p.} Q(G;\bar{X})$ iff $P(G;\bar{X}) \preceq_{coeff} Q(G;\bar{X})$.*

If $P(G;\bar{X}) \preceq_{coeff} Q(G;\bar{X})$ then $P(G;\bar{X}) \preceq_{subst} Q(G;\bar{X})$, but not conversely.

Most graph polynomials studied in the literature have several equivalent definitions:

1. by counting generalized colorings;
2. by generating functions;
3. by subset expansion formulas;
4. by recurrence relations;

5. by counting (weighted) homomorphisms.

We shall see that, by imposing SOL-definability,(1)-(3) give the same class of graph polynomials, whereas (4) and (5) are special cases thereof.

2.3.1 Complexity of the classical graph polynomials?

There are various problems with measuring the complexity of a multivariate graph polynomial $P(G; \bar{X})$:

Turing complexity:

Evaluation: Fix $x_0 \in \mathbb{Q}^m$. Measure the complexity of computing $P(G; x_0)$ as a function of the size of G.

Computing the coefficients: Measure the complexity of computing the coefficients of $P(G; x_0)$ as a function of the size of G.

It is usually in EXPTIME, often $\sharp$ P-complete, but sometimes in P-time. The Turing model does not fit the algebraic character of the problem.

BSS complexity: Think of a (weighted) graph being given by its adjacency matrix M_G. Measure the complexity of computing the coefficients of $P(G; \bar{X})$ from the matrix M_G.

It is usually in EXPTIME, but no convincing complexity classes fit the framework.

2.4 The matching polynomials: case study

We illustrate these concepts with the *bivariate matching polynomial*, as introduced by O.J. Heilman and E.J. Lieb, 1972.

The acyclic polynomial has important applications in Chemistry and molecular Physics of Ferromagnetisms. It was first studied in the 1970s (Heilman and Lieb; Kunz). See [18, 22, 10].

2.4.1 Two univariate matching polynomials

Recall the two univariate matching polynomials, (3) and (2) introduced in Section 1.3.1 and Section 1.4, together with the properties (4). From Section 1.10.2 we note that $gm(G; X) \preceq_{coeff} dm(G; X)$ and hence $gm(G; X) \preceq_{d.p.} dm(G; X)$. However, we do not have $dm(G; X) \preceq_{coeff} gm(G; X)$ since in order to reduce the coefficient computation for $dm(G; X)$ to that for $gm(G; X)$ requires knowing the value of $|V(G)| \preceq_{d.p.} dm(G; X)$, which is not available from $gm(G; X)$. Indeed, we saw earlier that it is not the case that $dm(G; X) \preceq_{d.p.} gm(G; X)$, as some graphs distinguished by $dm(G; X)$ are not distinguished by $gm(G; X)$ because, for example, $gm(G; X)$ does not detect isolated vertices. On the other hand $dm(G; X) \sim_{s.d.p.} gm(G; X)$, as the two matching polynomials are interreducible on graphs with the same number of vertices.

We denote by $m_k(G)$ the number of k-matchings of a graph G. The two matching polynomials are special cases of the *bivariate matching polynomial*

$$M(G,X,Y) = \sum_{k=0}^{\lfloor n/2 \rfloor} X^{n-2k} Y^k m_k(G) = \sum_A X^{|V(G)|-2|A|} Y^{|A|}$$

where A ranges over all subsets of $E(G)$ that are matchings of G ($|A|$ is the size of the matching A, and $|V(G)|-2|A|$ is the number of vertices not incident with any edge in A).

$M(G,X,Y) = \sum_{k=0}^{\lfloor n/2 \rfloor} X^{n-2k} Y^k m_k(G)$ can be viewed as a generating function. $M(G,X,Y) = \sum_A X^{|V(G)|-2|A|} Y^{|A|}$ can be viewed as a subset expansion.

Now we have $dm(G;X) = M(G;X,-1)$ and $gm(G;X) = M(G;1,X)$. In other words, both $dm(G;X)$ and $gm(G;X)$ are substitution instances of $M(G;X,Y)$.

2.4.2 The bivariate matching polynomial as a generalized chromatic polynomial

We want to show that the bivariate matching polynomial can be obtained in our framework. We use

- two sorts of colors $[k_1]$ and $[k_2]$;
- a coloring property consisting of 5-tuples $\langle V, [k_1], [k_2]; E, r_1, r_2 \rangle$, with two coloring relations $r_1 \subseteq E \times [k_1]$ and $r_2 \subseteq V \times [k_2]$ such that
  1. $r_1 \subseteq E \times [k_1]$ is a partial function, the domain of which is a matching M of G;
  2. and $r_2 \subseteq V \times [k_2]$ is a partial function, the domain of which is the set of vertices not covered by M.

2.5 Second-order logic

Second-order logic (SOL) is the natural language to talk about graph properties.

We shall show this informally and only after that define the syntax and semantics of SOL. We shall see we can also use SOL to define graph parameters.

Atomic formulas for graphs are $E(u,v)$ and $u = v$ for individual variables u, v, and $R(u_1, \ldots, u_m)$ for m-ary relation variables R. Examples of SOL fragments include (see Section 3.2 for further details):

- First-order logic (FOL):
  Closed under boolean operations and quantification over individual variables. No relation variables.
- Second-order logic (SOL):
  Closed under boolean operations and quantification over individual and relation variables of arbitrary but fixed arity.
- Monadic second-order logic (MSOL):
  Closed under boolean operations and quantification over individual and unary relation variables.

2.6 Graph properties

Regularity A graph G is (give definition in SOL):

- of degree bounded by $d \in \mathbb{N}$. (Every vertex has at most d neighbors.)
- k-regular ($k \in \mathbb{N}$). (Every vertex has exactly k neighbors.)
- regular. (Every vertex has exactly the same number of neighbors.)
- Regular and degree bounded by d.

Some regularity properties can be expressed in FOL using the following formulas:

- The vertices $v_0, v_1, \ldots, v_n$ are all different:

$$\mathrm{Diff}(v_0, v_1, \ldots, v_n) : \left(\bigwedge_{i=0,j=1,i<j}^{i,j \leq n} v_i \neq v_j \right)$$

- A vertex v_0 has degree at most d:

$$\mathrm{Deg}_{\leq d}(v_0) : \forall v_1, \ldots, v_d, v_{d+1} \left(\bigwedge_{i=0}^{d+1} E(v_0, v_i) \to \bigvee_{i=0,j=0,i\neq j}^{i=d+1,j=d+1} v_i = v_j \right)$$

- A vertex v_0 has degree at least d:

$$\mathrm{Deg}_{\geq d}(v_0) : \exists v_1, \ldots, v_d \left(\mathrm{Diff}(v_1, \ldots, v_d) \wedge \bigwedge_{i=1}^{d} E(v_0, v_i) \right)$$

In particular, we can define in FOL the following:

- k-regular;
- regular and of bounded degree d;

However, the following are *not* definable in FOL (nor in MSOL):

- regular;
- each vertex has even degree.

To show non-definability in FOL we need the machinery of *Ehrenfeucht-Fraïssé games* or *connection matrices*. See Section 3.

The following are definable in SOL:

- Two sets $A, B \subseteq V$ have the same size:

$$\mathrm{EQS}(A, B) : \exists R \left(\mathrm{Funct}(R, A, B) \wedge \mathrm{Inj}(R) \wedge \mathrm{Surj}(R) \right)$$

  where $\mathrm{Funct}(R, A, B), \mathrm{Inj}(R), \mathrm{Surj}(R)$ are FOL-formulas saying that R is a function from A to B which is one-to-one (injective) and onto (surjective).
- A vertex v has even degree:
  The set of neighbors of v can be partitioned into two sets of equal size

$$\mathrm{EDeg}(v_0) : \exists A, B \left(\mathrm{Part}(N_v, A, B) \wedge \mathrm{EQS}(A, B) \right).$$

- Two vertices u, v have the same degree:
  The set of neighbors N_u, N_v of u and v have the same size.

$$\mathrm{SDeg}(u, v) : \mathrm{EQS}(N_u, N_v).$$

Closure properties of graph classes. A graph property is called

- *hereditary* if it is closed under induced subgraphs.
- *monotone* if it is closed under subgraphs, not necessarily induced.
- *monotone decrasing* if it is closed under deletion of edges, but not necessarily of vertices.
- *monotone increasing* if it is closed under addition of edges, but not necessarily of vertices.
- *additive* if it is closed under disjoint unions.

Note that monotone implies hereditary and monotone decreasing.

Examples of closure properties:

- Regular graphs are only additive.
- Graphs of bounded degree d are monotone and additive.
- Cliques (complete graphs) are hereditary but not monotone.
- Connectivity is only monotone increasing.
- **Exercise:** Check the above closure properties of graph properties.
- **Exercise:** Check the above closure properties of all the graph properties discussed in the sequel.

Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs.

- We denote by $\mathrm{Forb}_{sub}(\mathcal{H})$ ($\mathrm{Forb}_{ind}(\mathcal{H})$) the class of graphs which have no (induced) subgraph isomorphic to some graph $H \in \mathcal{H}$.
- $\mathrm{Forb}_{sub}(\mathcal{H})$ is monotone and $\mathrm{Forb}_{ind}(\mathcal{H})$ is hereditary.

Theorem 2.6 (Exercise). *Let $\mathcal{P}$ be a monotone (hereditary) graph property. Then there exists a family $\mathcal{H} = \{H_i : i \in I\}$ of finite graphs such that $\mathcal{P} = \mathrm{Forb}_{sub}(\mathcal{H})$ (respectively $\mathcal{P} = \mathrm{Forb}_{ind}(\mathcal{H})$).*

Proposition 2.7. *Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs with I finite. Then both $\mathrm{Forb}_{sub}(\mathcal{H})$ and $\mathrm{Forb}_{ind}(\mathcal{H})$ are definable in* FOL.

Homework 1 Characterize the following graph properties using $\mathrm{Forb}_{sub}(\mathcal{H})$ or $\mathrm{Forb}_{ind}(\mathcal{H})$, and determine their definability in FOL and SOL:

- Forests
- Cliques
- Find other examples! You might wish to consult [4].

Colorability. Let $\mathcal{P}$ be a graph property. A graph G is (give definition in SOL, MSOL):

- 3-colorable:
  The vertices of G can be partitioned into three disjoint sets $C_i : i = 1, 2, 3$, such that the induced graphs $G[C_i]$ consist only of isolated points.
  This can be expressed in MSOL.
- k-$\mathcal{P}$-colorable ($k \in \mathbb{N}$):
  The vertices of G can be partitioned into k disjoint sets $C_i : i = 1, \ldots, k$, such that the induced graphs $G[C_i]$ are in $\mathcal{P}$.
  If $\mathcal{P}$ is definable in SOL (MSOL), this is also definable in SOL (MSOL).

- $\mathcal{P}$-colorable:
  The vertices of G can be partitioned into disjoint sets $C_i : i \in I \subset \mathbb{N}$, such that the induced graphs $G[C_i]$ are in $\mathcal{P}$.
  This is definable in SOL provided $\mathcal{P}$ is. It is not MSOL-definable.

A subset V_1 of vertices of a graph $G = (V, E)$ is independent if it induces a graph of isolated points (without neighbors or loops). A graph is k-colorable if its vertices can be partitioned into k independent sets. Let

$$\begin{gathered}\mathrm{Part}(X_1, X_2, X_3) : \\ ((X_1 \cup X_2 \cup X_3 = V) \wedge ((X_1 \cap X_2) = (X_2 \cap X_3) = (X_3 \cap X_1) = \emptyset))\end{gathered}$$

and

$$\mathrm{Ind}(X) : \ (\forall v_1 \in X)(\forall v_2 \in X)\neg E(v_1, v_2)$$

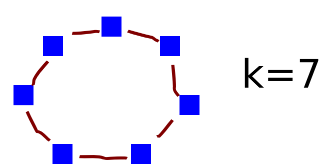
With this 3-colorable can be expressed as

$$\exists C_1 \exists C_2 \exists C_3 \, (\mathrm{Part}(C_1, C_2, C_3) \wedge \mathrm{Ind}(C_1) \wedge \mathrm{Ind}(C_2) \wedge \mathrm{Ind}(C_3))$$

We have expressed 3-colorability by a formula in MSOL.

Question: Can we express this in FOL?

Chordality. A graph is a simple cycle of length k of it is of the form:



A graph is a simple cycle iff it is connected and 2-regular. A graph G is *chordal* (or *triangulated*) if there is no induced subgraph of G isomorphic to a simple cycle of length ≥ 4.

Exercise: Find a MSOL-expression for chordality.

Eulerian graphs. A graph $G = (V, E)$ is *Eulerian* if we can follow each edge exactly once, pass through all the edges, and return to the point of departure.

Equivalently: Can we order all the edges of E $e_1, e_2, e_3, \ldots e_m$ and choose beginning and end of the edge $e_i = (u_i, v_i)$ such that for all i, $v_i = u_{i+1}$ and $v_m = u_1$.

$$\begin{gathered}\exists R \, (\mathrm{LinOrd}(R, E) \wedge \\ (\forall u, v, u', v' \mathrm{First}(R, u, v) \wedge \mathrm{Last}(R, u', v') \to u = v') \wedge \\ (\forall u, v, u', v' \mathrm{Next}(R, u, v, u'v') \to v = u'))\end{gathered}$$

with the obvious meaning of $\mathrm{LinOrd}(R, E)$, $\mathrm{First}(R, u, v)$ and $\mathrm{Last}(u, v)$. This allows us to express the Eulerian property in SOL.

Alternatively, we can use

Theorem (Euler): A graph is Eulerian iff it is connected and each vertex has even degree.

As we shall see later, being Eulerian cannot be expressed in MSOL.

Hamiltonian graphs. A graph with n vertices is Hamiltonian if it contains a spanning subgraph that is a cycle of size n.

Given the formulas $\mathrm{Conn}(V_1, E_1)$: (V_1, E_1) is connected, and $\mathrm{Cycle}(V_1, E_1)$: (V_1, E_1) is a cycle, i.e., regular of degree 2 and connected, Hamiltonicity can be expressed in SOL by $\mathrm{Ham}(V, E) : \exists V_1 \exists E_1 \left(Cycle(V_1, E_1) \wedge E_1 \subseteq E \wedge V_1 = V \right).$

A subtle point: graphs versus hypergraphs.

- Graphs are structures with a universe V of vertices, and a binary edge relation E. There can be at most one edge between two vertices.
- Hypergraphs have as their universe two disjoint sets V and E and an incidence (hyperedge) relation $R(u, v, e)$. There can be many edges between two vertices.
- In both cases the relations are symmetric in the vertices.
- A graph G can be viewed as a hypergraph (h-graph) $h(G)$ in which there is at most one edge (up to symmetry) between two vertices. See Figure 1.
- There is a one-to-one correspondence between graph and h-graphs.
- FOL and SOL are equally expressive on graphs and h-graphs.
- MSOL is more expressive on h-graphs than on graphs.
  Hamiltonicity is not definable in MSOL on graphs, but is definable on h-graphs.

We shall discuss this in detail in a later lecture.

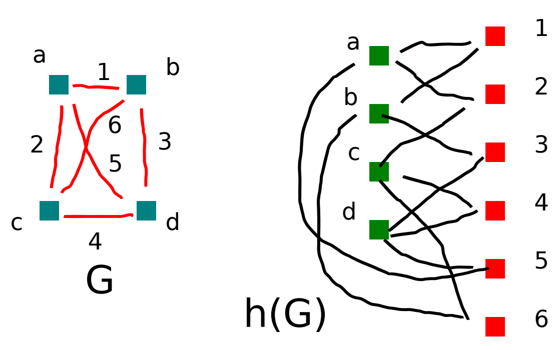
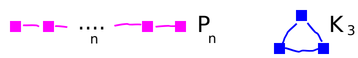


Figure 1: G and $h(G)$

How to prove definablity in SOL, MSOL and FOL? So far we have looked at properties of abstract (directed) graphs and hypergraphs.

- Formulate the property using set-theoretic language of finite sets over the set of vertices and edges and their incidence relation.
- Try to mimic this formulation in SOL.
- If you succeed, try to do it in MSOL or even FOL.

Test your fluency in SOL! (Homework).



Express the following properties in FOL, if possible.

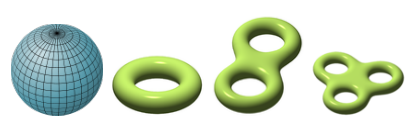
- A graph G is a cograph if and only if there is no induced subgraph of G isomorphic to a P_4.
- A G is P_4-sparse if no set of 5 vertices induce more than one P_4 in G.
- Triangle-free graphs: there is no induced K_3.
- Existence of a prescribed (induced) subgraph H.
- H-free graphs: non-existence of a prescribed (induced) subgraph H.
- $\mathcal{P}$-free graphs (for a graph property $\mathcal{P}$): non-existence of an induced subgraph $H \in \mathcal{P}$.

Topological properties of graphs (from Wikipedia) [3]

So far our graph properties have been formulated in the language of graphs, involving as basic concepts only vertices, edges and their incidence relations. Topological graph theory studies the embedding of graphs in surfaces, spatial embeddings of graphs, and graphs as topological spaces.

- A graph is planar if it is isomorphic to a plane graph.
- The genus of a graph is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles (i.e. an oriented surface of genus n).

  Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing.



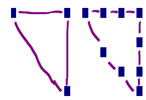
genus: $0, 1, 2, 3$

Planar graphs. A graph is planar iff it is isomorphic to a plane graph. This definition involves the geometry of th Euclidean plane.

How can we express planarity *without* geometry ?

A subdivision of a graph G is a graph formed by subdividing its edges into paths of one or more edges.

[3] `http://en.wikipedia.org/wiki/Genus_(mathematics)`



K_3 and a subdivision of K_3

Theorem 2.8 (Kuratowski's Theorem [4]). *A finite graph G is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$.*

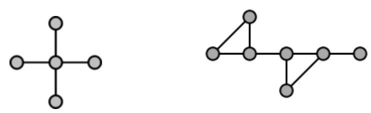
Theorem 2.9. *Planarity is definable in* MSOL.

- We use Kuratowski's Theorem.
- For a fixed graph H, *G is a subdivision of H* is definable in MSOL.
- For a graph property $\mathcal{P}$ definable in MSOL, *G has a subgraph $H \in \mathcal{P}$* is definable in MSOL.

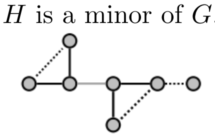
Exercise: Prove the last two statements.

Graph minors [5]

An undirected graph H is called a *minor* of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges.



H is a minor of G.



First construct a subgraph of G by deleting the dashed edges (and the resulting isolated vertex); then contract the thin edge (merging the two vertices it connects).

Proposition 2.10. *For fixed H, the statement that* H is a minor of G *is definable in* MSOL.

- Edge contraction is an operation which removes an edge from a graph while simultaneously merging the two vertices it used to connect.
- An undirected graph H is a minor of another undirected graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices.
- The order in which a sequence of such contractions and deletions is performed on G does not affect the resulting graph H.
- Let $V(H) = \{v_1, \ldots, v_m\}$. We have to find $V_1, \ldots, V_m \subseteq V(G)$ which we all contract to a vertex u_i corresponding to v_i such that V_i connects to V_j iff $(v_i, v_j) \in E(H)$.
- The vertices in $V(G) - \bigcup_i^m V_i$ are discarded.

[4] From `http://en.wikipedia.org/wiki/Kuratowski's_theorem`.
[5] From `http://en.wikipedia.org/wiki/Graph_minor`.

Minor closed graph classes

- H is a topological minor of G if G has a subgraph which is isomorphic to a subdivision of H.
- A graph property $\mathcal{P}$ is closed under (topological) minors, if whenever $G \in \mathcal{P}$ and H is a (topological) minor of G then also $H \in \mathcal{P}$.

Examples:

- Trees are not closed under minors, but forests are.
- Graphs of degree at most 2 are minor closed, but graphs of degree at most 3 are not.
- Planar graphs are both closed under minors and topological minors.

Forbidden minors. Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs.

- We denote by $\mathrm{Forb}_{min}(\mathcal{H})$ ($\mathrm{Forb}_{tmin}(\mathcal{H})$) the class of graphs G which have no (topological) minors isomorphic to some graph $H \in \mathcal{H}$.
- $\mathrm{Forb}_{min}(\mathcal{H})$ is closed under topological minors and monotone, and hence it is hereditary.

Theorem 2.11 (Exercise). *Let $\mathcal{P}$ be a graph property closed under (topological) minors. Then there exists a family $\mathcal{H} = \{H_i : i \in I\}$ of finite graphs such that $\mathcal{P} = \mathrm{Forb}_{min}(\mathcal{H})$ (respectively $\mathcal{P} = \mathrm{Forb}_{tmin}(\mathcal{H})$).*

Proposition 2.12. *Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of graphs with I finite. Then both $\mathrm{Forb}_{min}(\mathcal{H})$ and $\mathrm{Forb}_{tmin}(\mathcal{H})$ are definable in* MSOL.

Here is one of the deepest theorems in structural graph theory:

Theorem 2.13 (The Graph Minor Theorem (aka Robertson-Seymour Theorem))**.** *Let $\mathcal{P}$ be a graph property closed under minors. Then $\mathcal{P} = \mathrm{Forb}_{min}(\mathcal{H})$ for some finite $\mathcal{H}$.*

Corollary 2.14. *Every graph property $\mathcal{P}$ property closed under minors is definable in* MSOL.

The following theorem gives another proof that planarity is MSOL-definable.

Theorem 2.15 (Wagner's Theorem)**.** *A graph G is planar iff K_5 and $K_{3,3}$ are not minors of G.*

Conjecture 2.16 (Hadwiger's Conjecture)**.** *If a graph G is not k-colorable then it has the complete graph K_{k+1} as a minor.*

The conjecture has been proven for $k \leq 6$. The converse is not true. There are bipartite graphs with a K_4 minor.

2.7 Logic and complexity: regular languages

Let $L \subseteq \Sigma^\star$ be a *language*, i.e., a set of words over the alphabet Σ.

We assume you are familiar with automata theory!

Theorem 2.17 (Kleene; Büchi, Elgot; Trakhtenbrot)**.** *The following are equivalent:*

- *L is recognizable by a deterministic finite automaton.*
- *L is recognizable by a non-deterministic finite automaton.*
- *L is regular, i.e., describable by a regular expression*
- *The set of τ_{word}-structures $\mathfrak{A}_w$ with $w \in L$ is definable in* $\mathrm{MSOL}(\tau_{word})$*.*

We need to recall some complexity classes:

L: Deterministic logarithmic space.

NL: Non-deterministic logarithmic space.

P: Deterministic polynomial time.

NP: Non-deterministic polynomial time.

PH: The polynomial hierarchy.

♯P: Counting predicates in **P** (Valiant's class)

PSpace: Deterministic polynomial space.

Some results on the complexity of SOL properties.

- (Fagin, Christen)

  The **NP**-properties of classes of τ-structures are exactly the $\exists$SOL-definable properties.

- (Meyer, Stockmeyer) The **PH**-properties (in the *polynomial hierarchy*) of classes of τ-structures are exactly the SOL-definable properties.

- (Makowsky, Pnueli:) For every level Σ_n^P of **PH** there are MSOL-definable classes which are complete for it.

We have

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PH} \subseteq \sharp\mathbf{P} \subseteq \mathbf{PSpace}$$

- To show that **PH** does not collapse to **NP** we have to find a τ-sentence $\phi \in \mathrm{SOL}(\tau)$ which is not equivalent over finite structures to an existential τ-sentence $\psi \in \mathrm{SOL}(\tau)$.
- Every sentence $\phi \in \mathrm{SOL}(\tau)$ is equivalent (over finite structures) to an existential sentence $\psi \in \mathrm{SOL}(\tau)$ iff $\mathbf{NP} = \mathbf{CoNP}$. Note we allow arbitrary arities of the quantified relation variables. Over infinite structures this is known to be false (Rabin).
- If there is a $\phi \in \mathrm{SOL}(\tau)$ which is not equivalent to an existential sentence, then $\mathbf{P} \neq \mathbf{NP}$. And there should be such a sentence!
- To show that **PSpace** is different from **PH** it suffices to find a **PSpace**-complete graph property which is not SOL-definable.

2.8 HEX, geography, and Shannon switching

- The game HEX:[6] Given a graph G and two vertices s, t. Players I and II color alternately vertices in $V - \{s, t\}$ white and black respectively. Player I tries to construct a white path from s to t and Player II tries to prevent this.
  HEX: The class of graphs which allow a winning strategy for Player I.
- The game GEOGRAPHY: Given a directed graph G, Players I and II choose alternately new edges starting at an endpoint of the edge chosen last. The first who cannot find such an edge loses.
  GEO: The class of graphs which allow a winning strategy for Player I.

Theorem (Even, Tarjan) HEX is **PSPACE**-complete.

Theorem (Schaefer) GEO is **PSPACE**-complete.

Problem Are they SOL-definable?

This would imply that PSPACE = PH, and the polynomial hierarchy collapses to some finite level!

Short versions Fix $k \in \mathbb{N}$. SHORT-HEX, SHORT-GEOGRAPHY asks whether Player I can win in k moves.

S-HEX and S-GEO are the class of (ordered) graphs in which Player I has a winning strategy.

S-HEX and S-GEO are FOL-definable for fixed k (and therefore solvable in **P**).

The game of Shannon switching. Given a graph G and two vertices s, t. Players I and II color alternately edges in E white and black respectively.Player II tries to construct a white path from s to t and Player I tries to prevent this.

ShaSwi: The class of graphs which allow a winning strategy for Player II.

Theorem 2.18 (A. Lehmann, 1964, [17])**.** ShaSwi *is* SOL-*definable.*

Proof. The Shannon Switching game is winning for Player II if and only if the graph contains two edge-disjoint trees on a common subset of vertices that contains the two distinguished vertices. □

Corollary 2.19. ShaSwi *is in* **PH***.*

Challenge. Show that: HEX, GEOGRAPHY, and ShaSwi are NOT definable in MSOL! This may just be achievable with the techniques of the next lecture; see Section 3.

[6] More on HEX at `http://maarup.net/thomas/hex/`.

2.9 The role of order

Let $\tau_=$ be the one-sorted vocabulary without any relation or constant symbols. We have only equality as atomic formulas.

Let $\tau_<$ be the one-sorted vocabulary with one binary relation symbol $R_<$, which will be interpreted as a linear order.

- The class of structures of even cardinality EVEN is not definable in MSOL($\tau_=$). We shall prove this later.
- The class of structures of even cardinality EVEN is definable in MSOL($\tau_<$) with order by a formula ϕ_{EVEN}.

Constructing ϕ_{EVEN}. We use the order to define the binary relation 2NEXT and the unary relation Odd

- For a structure $\mathfrak{A} = \langle A, < \rangle$, let $(a, b) \in \text{2NEXT}^{\mathfrak{A}}$ iff $a < b$ and there is exactly one element strictly between a and b.
- The first element is in $\text{Odd}^{\mathfrak{A}}$.
  If $a \in \text{Odd}^{\mathfrak{A}}$ and $(a, b) \in \text{2NEXT}^{\mathfrak{A}}$ then $b \in \text{Odd}^{\mathfrak{A}}$.
- Let ϕ_{EVEN} be the formula which says that the last element is not in Odd.
- Now the a structure $\langle A, < \rangle$ is in EVEN iff its last element is not in $\text{Odd}^{\mathfrak{A}}$.

This constructs ϕ_{EVEN} $\square$

Order-invariance of ϕ_{EVEN}. In the previous example EVEN, the MSOL($\tau_<$)-formula ϕ_{EVEN} is *order-invariant* in the following sense:

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two structures with universe A and different order relations $<_1$ and $<_2$. Then $\mathfrak{A}_1 \models \phi_{EVEN}$ iff $\mathfrak{A}_2 \models \phi_{EVEN}$.

We generalize this. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two $\tau \cup \{R_<\}$-structures with universe A and different order relations $\mathfrak{A}_1(R_<) = <_1$ and $\mathfrak{A}_2(R_<) = <_2$ but for all other symbols in $R \in \tau$ we have $\mathfrak{A}_1(R) = \mathfrak{A}_2(R)$.

A $\tau \cup \{R_<\}$-formula in SOL is order-invariant if for all structures $\mathfrak{A}_1, \mathfrak{A}_2$ as above we have

$$\mathfrak{A}_1 \models \phi \text{ iff } \mathfrak{A}_2 \models \phi$$

The fragment HornESOL(τ)**.**

- A quantifier-free τ-formula is a Horn clause if it is a disjunction of atomic or negated atomic formulas where at most one is not negated.

$$\neg\alpha_1 \vee \neg\alpha_2 \vee \ldots \vee \neg\alpha_n \vee \beta$$

  where α_i, β are atomic.
- A quantifier-free τ-formula is a Horn formula if it is a conjunction of Horn clauses.
- A formula $\phi \in \text{SOL}(\tau)$ is in HornESOL(τ) if it is of the form

$$\exists U_{1,r_1}, U_{2,r_2}, \ldots, U_{k,r_k} \forall v_1, \ldots, v_m H(v_1, \ldots, v_m, U_{1,r_1}, U_{2,r_2}, \ldots, U_{k,r_k})$$

  where H is a Horn formula and v_i are first-order variables.

Some classes of graphs order-invariantly (o.i.) definable in HornESOL(τ_{graph}) are the following.

- Graphs of even cardinality, of even degree. Order is needed!
- Bipartite graphs $G = (V_1, V_2, E)$ with $|V_1| = |V_2|$.
- Regular graphs, and regular graphs of even degree.
- Connected graphs.
- Eulerian graphs.

The Immermann-Vardi-Graedel Theorem (IVG) Let τ be a relational vocabulary with a binary relation for the ordering of the universe.

Theorem 2.20 (Immermann, Vardi, Graedel, 1980-4)**.** *Let $\mathcal{C}$ be a set of finite τ-structures. The following are equivalent:*

- $\mathcal{C} \in \mathbf{P}$*;*
- *there is a τ-formula $\phi \in$ HornESOL(τ) such that $\mathfrak{A} \in \mathcal{C}$ iff $\mathfrak{A} \models \phi$.*

Here the presence of the ordering is crucial: without it the class of structures for the empty vocabulary of even cardinality is in $\mathbf{P}$, but not definable in HornESOL.

We can also obtain an order-invariant version of Theorem 2.20. Let τ be a relational vocabulary and $\tau_1 = \tau \cup \{R_<\}$. with a binary relation for the ordering of the universe.

Theorem 2.21 (Graedel, 1980-4; Dawar, Makowsky)**.** *Let $\mathcal{C}$ be a set of finite τ-structures. The following are equivalent:*

- $\mathcal{C} \in \mathbf{P}$*;*
- *there is an order-invariant τ_1-formula $\phi \in$ HornESOL(τ) such that, for all τ-structures $\mathfrak{A}$ and linear orderings $R^A \subset \mathfrak{A}(V)^2$, $\mathfrak{A} \in \mathcal{C}$ iff $\langle \mathfrak{A}, R^A \rangle \models \phi$.*

Conclusion: the logical equivalent to $\mathbf{P} = \mathbf{NP}$. Let τ be a relational vocabulary which contains a binary relation for the ordering of the universe. The following are equivalent:

- $\mathbf{P} = \mathbf{NP}$ in the classical framework.
- Every ESOL(τ)-formula is equivalent over finite ordered τ-structures to some HornESOL(τ)-formula.
- Every o.i. ESOL(τ)-formula is equivalent over finite ordered τ-structures to some o.i. HornESOL(τ)-formula.

Logics capturing complexity classes. Without requiring the presence of order we have:

- A class $\mathcal{C}$ of finite structures is in $\mathbf{NP}$ iff $\mathcal{C}$ is definable in existential SOL.
- A class $\mathcal{C}$ of finite structures is in $\mathbf{PH}$ iff $\mathcal{C}$ is definable in SOL.

By requiring the presence of an order relation we have

- A class $\mathcal{C}$ of finite structures is in $\mathbf{P}$ iff $\mathcal{C}$ is order-invariantly definable in existential HornESOL.
- There are similar theorems for $\mathbf{L}, \mathbf{NL}, \mathbf{PSpace}$.

3 Hankel matrices, connection matrices, and definability of graph invariants

- Hankel matrices
- Logics and definability of numeric graph invariants.
- Non-definability via complexity theory.
- Typical properties of graph parameters.
- Connection matrices (aka Hankel matrices) and their rank, I.
- Connection matrices (aka Hankel matrices) and their rank, II.
- The Finite Rank Theorem (FRT).
- Applications of FRT for graph properties.
- Applications of FRT for graph polynomials.
- Merits and limitations of FRT.

CMSOL**-definable graph parameters** CMSOL is monadic second-order logic augmented by modular counting quantifiers.

- I have developed with various co-authors a framework of definability of numeric graph parameters. B. Courcelle, B. Godlin, T. Kotek, E. Ravve
- In this talk we discuss a method of proving non-definability in CMSOL of numeric graph parameters which take values in a field.
- The CMSOL-definable graph parameters behave similarly to CMSOL-definable graph properties.
    1. On graphs of bounded width they are in **FPT**, where the notion of width and the the notion of monadic quantification have to fit correspondingly.
    2. All classical graph polynomials (Tutte polynomial, matching polynomial, chromatic polynomial, interlace polynomial) and many more are CMSOL-definable using order on vertices in an invariant way
    3. On recursively defined graph sequences (like P_n, C_n, L_n, etc) they can be computed via linear recurrence relations.

3.1 Hankel aka connection matrices

Hankel matrices (over a field $\mathcal{F}$) Let $f : \mathcal{F} \to \mathcal{F}$ be a function over a field $\mathcal{F}$. A finite or infinite matrix $H(f) = h_{i,j}$ is a Hankel matrix for f if $H_{i,j} = f(i + j)$. Hankel matrices have many applications in numeric analysis, probability theory and combinatorics:

- Padé approximations
- Orthogonal polynomials
- Theory of moments in probability theory
- Coding theory (BCH codes, Berlekamp-Massey algorithm)
- Combinatorial enumeration (lattice paths, Young tableaux, matching theory)

Hankel matrices over words Let Σ be a finite alphabet, $\mathcal{F}$ a field, and $f : \Sigma^\star \to \mathcal{F}$ a function on words. A finite or infinite matrix $H(f) = h_{u,v}$ indexed by words $u, v \in \Sigma^\star$ is a Hankel matrix for f if $h_{u,v} = f(u \circ v)$. Here $\circ$ denotes concatenation.

Hankel matrices over words have applications in formal language theory and stochastic automata (J. Carlyle and A. Paz 1971), learning theory (exact learning of queries) (A.Beimel, F. Bergadano, N. Bshouty, E. Kushilevitz, S. Varricchio 1998, J. Oncina 2008), and definability of picture languages (O. Matz 1998, and D. Giammarresi and A. Restivo 2008).

Hankel matrices for graphs If we want to define Hankel matrices for (labeled) graphs, what plays the role of concatenation?

- Disjoint union. Used by Freedman, Lovász and Schrijver (2007) for characterizing multiplicative graph parameters over the real numbers
- k-unions (connections, connection matrices). Used by Freedman, Lovász, Schrijver and Szegedy (2007+), for characterizing various forms of partition functions.
- Joins, cartesian products, generalized sum-like operations. Used by Godlin, Kotek and Makowsky (2008) to prove non-definability.

3.2 Logics

Let $\mathcal{L}$ be a subset of SOL. $\mathcal{L}$ is a fragment of SOL if the following conditions hold:

1. For every finite relational vocabulary τ the set of $\mathcal{L}(\tau)$ formulas contains all the atomic τ-formulas and is closed under boolean operations and renaming of relation and constant symbols.
2. $\mathcal{L}$ is equipped with a notion of quantifier rank and we denote by $\mathcal{L}_q(\tau)$ the set of formulas of quantifier rank at most q. Quantifier rank is subadditive under substitution of subformulas,
3. The set of formulas of $\mathcal{L}_q(\tau)$ with a fixed set of free variables is, up to logical equivalence, finite.
4. Furthermore, if $\phi(x)$ is a formula of $\mathcal{L}_q(\tau)$ with x a free variable of $\mathcal{L}$, then there is a formula ψ logically equivalent to $\exists x \phi(x)$ in $\mathcal{L}_{q'}(\tau)$ with $q' \geq q + 1$.
5. A fragment of SOL is called *tame* if it is closed under scalar transductions.[7]

Some typical fragments are:

- FOL.
- MSOL.
- Logics augmented by modular counting quantifiers: $D_{m,i} x \phi(x)$ which says that the numbers of elements satisfying ϕ equals i modulo m.
- CFOL, CMSOL denote the logics FOL, resp. MSOL, augmented by all the modular counting quantifiers.

[7] *Editors' note.* For *transductions* see the examples in Section 3.8 below and [5, chapter 7]. (A preprint of the book [5] is available at `http://www.labri.fr/perso/courcell/Book/TheBook.pdf`.)

- Logics augmented by Lindström quantifiers.
- Logics restricted to a fixed finite set of bound or free variables.

3.3 Definability of numeric graph invariants and graph polynomials

We denote by $G = (V(G), E(G))$ a graph, and by $\mathcal{G}$ the class of finite graphs. A *numeric graph invariant* or *graph parameter* is a function

$$f : \mathcal{G} \to \mathbb{R}$$

which is invariant under graph isomorphism. Some examples:

1. Cardinalities: $|V(G)|$, $|E(G)|$
2. Counting configurations:
   $k(G)$ the number of connected components, $m_k(G)$ the number of k-matchings
3. Size of configurations:
   $\omega(G)$ the clique number, $\chi(G)$ the chromatic number.
4. Evaluations of graph polynomials:
   $\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$. $T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ for any $(x, y) \in \mathbb{R}^2$.

Let $\mathcal{R}$ be a (polynomial) ring. A graph parameter $f : \mathcal{G} \to \mathcal{R}$ is *$\mathcal{L}$-definable* if it can be defined inductively as follows:

- Monomials are of the form $\prod_{\bar{v}:\phi(\bar{v})} t$ where t is an element of the ring $\mathcal{R}$ and ϕ is a formula in $\mathcal{L}$ with first-order variables $\bar{v}$.
- Polynomials are obtained by closing under small products, small sums and large sums.

"Small" here means polynomial-sized; see below and [19, section 5.2]. Usually, summation is allowed over second-order variables, whereas products are over first-order variables. $\mathcal{L}$ is typically MSOL or a suitable fragment thereof. We are especially interested in MSOL itself and CMSOL (monadic second-order logic augmented by modular counting quantifiers).

If $\mathcal{L}$ is SOL we denote the definable graph parameters by $\mathrm{SOLEVAL}_{\mathcal{R}}$, and similarly for MSOL and CMSOL. Our definition of SOLEVAL is reminiscent of Skolem's definition of the *lower elementary functions*.

How can we prove definability and non-definability of graph parameters in some logic $\mathcal{L}$? In particular:

- How to prove that $k(G)$ is not CFOL-definable?
- How to prove that $\omega(G)$ is not CMSOL-definable?
- How to prove that the chromatic number $\chi(G)$ or
  the chromatic polynomial $\chi(G, X)$ is not CMSOL-definable?

We first give examples of definability of numeric graph invariants and polynomials using small, i.e., polynomial-sized sums and products:

(i) The cardinality of V is FOL-definable by

$$\sum_{v \in V} 1$$

(ii) The number of connected components of a graph G, $k(G)$, is MSOL-definable by

$$\sum_{C \subseteq V : \text{component}(C)} 1$$

where component(C) says that C is a connected component.

(iii) The graph polynomial $X^{k(G)}$ is MSOL-definable by

$$\prod_{c \in V : \text{first}-\text{in}-\text{comp}(c)} X$$

if we have a linear order in the vertices and $\text{first} - \text{in} - \text{comp}(c)$ says that c is the first element in a connected component.

Now we give examples with possibly large sums, i.e., of exponential size:

(iv) The number of cliques in a graph is MSOL-definable by

$$\sum_{C \subseteq V : \text{clique}(C)} 1$$

where clique(C) says that C induces a complete graph.

(v) Similarly the number of maximal cliques is MSOL-definable by

$$\sum_{C \subseteq V : \text{maxclique}(C)} 1$$

where maxclique(C) says that C induces a maximal complete graph.

(vi) The clique number of G, $\omega(G)$, is is SOL-definable by

$$\sum_{C \subseteq V : \text{largest}-\text{clique}(C)} 1$$

where $\text{largest} - \text{clique}(C)$ says that C induces a maximal complete graph of largest size.

3.3.1 Graph properties

A *graph property* or *boolean graph invariant* is a function

$$f : \mathcal{G} \to \mathbb{Z}_2$$

which is invariant under graph isomorphism.

More traditionally, a graph property $\mathcal{P} = \mathcal{P}_f$ is a family of graphs closed under isomorphisms given by $\mathcal{P}_f = \{G : f(G) = 1\}$. Some examples are:

1. $\mathcal{P}$ is hereditary if it is closed under induced subgraphs.
2. $\mathcal{P}$ is monotone if it is closed under (not necessarily induced) subgraphs.
3. $\mathcal{P}$ is definable in some logic $\mathcal{L}$ if there is a formula $\phi \in \mathcal{L}$ such that $P = \{G : G \models \phi\}$.
4. Regular graphs of fixed degree d are definable in FOL.
5. Connectivity and planarity are definable in MSOL.

3.3.2 Non-definability via complexity assumptions

Harmonious colorings. Recall that a vertex coloring of a graph G with k colors is harmonious if it is proper and each pair of colors appears at most once along an edge. The harmonious index of a graph G is the smallest k such that there is a harmonious coloring with k colors.

- J.E. Hopcroft and M.S. Krishnamoorthy studied harmonious colorings in 1983.
- B. Courcelle, Makowsky and U. Rotics have shown that graph parameters (polynomials) definable in CMSOL can becomputed in polynomial time for graphs of tree-width at most k.
- K. Edwards and C. McDiarmid showed that computing the harmonious index is **NP**-hard even on trees.
- So assuming $\mathbf{P} \neq \mathbf{NP}$, the harmonious index is not CMSOL-definable, because trees have tree-width 1.

Chromaticity.

- B. Courcelle, J.A.M. and U. Rotics proved that graph parameters (polynomials) definable in CMSOL in the language of graphs can be computed in polynomial time for graphs of clique-width at most k.
- The Exponential Time Hypothesis (**ETH**) says that $3 - SAT$ cannot be solved in time $2^{o(n)}$. It was first formulated by R. Impagliazzo, R. Paturi and F. Zane in 2001.
- F. Fomin, P. Golovach, D. Lokshtanov and S. Saurabh proved that, assuming that **ETH** holds, the chromatic number $\chi(G)$ cannot be computed in polynomial time.
- Therefore, assuming **ETH**, the chromatic number and the chromatic polynomial are not CMSOL-definable.

There are many other non-definability results which can obtained like this, for example graph parameters derived from dominating sets or the size of a maximal cut. Our goal is to prove non-definability without complexity-theoretic assumptions.

3.4 Additive and multiplicative graph parameters with respect to a binary operation $\Box$

Let $G_1 \Box G_2$ denote a binary operation on graphs G_1 and G_2.A graph parameter f is *additive* if $f(G_1 \Box G_2) = f(G_1) + f(G_2)$ and *multiplicative* if $f(G_1 \Box G_2) = f(G_1) \cdot f(G_2)$.

For $\Box$ equal to *disjoint union* we have:

1. $|V(G)|, |E(G)|, k(G)$ are not multiplicative, but additive.
2. $k(G)$ and $b(G)$ are additive ($b(G)$ is the number of 2-connected components of G).
3. $\chi(G)$ and $\omega(G)$ are neither additive nor multiplicative.
4. The number of perfect matchings $pm(G)$ is multiplicative and so is the generating matching polynomial $\sum_k m_k(G) X^k$. Note that $m_k(G)$ is not

multiplicative.

5. The graph polynomials $\chi(G,\lambda)$ and $T(G,X,Y)$ are multiplicative.

A graph parameter f is *maximizing* if $f(G_1 \Box G_2) = \max\{f(G_1), f(G_2)\}$ and *minimizing* if $f(G_1 \Box G_2) = \min\{f(G_1), f(G_2)\}$. Again for $\Box$ equal to disjoint union we have

1. The various chromatic numbers $\chi(G)$, $\chi_e(G)$, $\chi_t(G)$ are maximizing.
2. The maximum clique size $\omega(G)$ and the maximum degree $\Delta(G)$ are maximizing.
3. The tree-width $tw(G)$ and the clique-width $cw(G)$ of a graph are maximizing.
4. The minimum degree $\delta(G)$ and the girth $g(G)$ are minimizing.

3.5 The connection matrix of a graph parameter with respect to disjoint union $\sqcup$

Let G_i be an enumeration of all finite graphs (up to isomorphism). The *(full) connection matrix* $M(f, \sqcup)$ is defined as the matrix with (i,j)-entry

$$m_{i,j}(f, \sqcup) := f(G_i \sqcup G_j).$$

The rank of $M(f, \sqcup)$ is denoted by $r(f, \sqcup)$. We shall often look at various infinite submatrices of the full connection matrix.

Examples: Check with $|V(G)|$ and $2^{|V(G)|}$.

Computing $r(f, \sqcup)$

Proposition 3.1.

1. *If f is multiplicative, $r(f, \sqcup) = 1$.*
2. *If f is additive, $r(f, \sqcup) = 2$.*
3. *If f is maximizing or minimizing, $r(f, \sqcup)$ is infinite.*
4. *For the average degree $d(G)$ of a graph, $r(d, \sqcup)$ is infinite.*

Proof. The first three statements are easy.

For $f = d$ we have

$$m_{i,j}(d, \sqcup) = 2\frac{|E(G_i)| + |E(G_j)|}{|V(G_i)| + |V(G_j)|}.$$

This contains, for graphs G_i, G_j with a fixed total number of edges e, the Cauchy matrix $(\frac{2e}{i+j})$, hence $r(d, \sqcup)$ is infinite. $\Box$

The next theorem characterizes multiplicative graph parameters.

Theorem 3.2 ([9] Proposition 2.1)**.** *Suppose f, g are graph parameters with values in an ordered field, and $g(G) \neq 0$ for some graph G. Then*

- *$f(G)$ is additive iff $g(G) = 2^{f(G)}$ is multiplicative;*
- *g is multiplicative iff $M(g, \sqcup)$ has rank 1 and is positive semi-definite.*

Recall: A finite square matrix M over an ordered field is *positive semi-definite* if for all vectors $\bar{x}$ we have $\bar{x}M\bar{x}^\top \geq 0$. An infinite matrix is positive semi-definite, if every finite principal submatrix is positive semi-definite.

3.6 General connection matrices (aka Hankel matrices)

Let $\mathcal{C}$ be a class of possibly labeled graphs, hypergraphs or τ-structures. Let $\Box$ be a binary operation defined on $\mathcal{C}$. Let G_i be an enumeration of all (labeled) finite graphs (structures) in $\mathcal{C}$. Let f be graph parameter (more generally, an invariant of the structures under consideration).

The *(full) connection matrix* $M(f, \Box)$ is defined by

$$M(f, \Box)_{i,j} = f(G_i \Box G_j).$$

We denote by $r(f, \Box)$ the rank of $M(f, \Box)$. We shall often look at infinite submatrices of $M(f, \Box)$.

To compute $r(f, \Box)$ we can use Proposition 3.1, replacing $\sqcup$ by $\Box$ in (1),(2) and (3). The same proof carries over.

3.7 $\mathcal{L}$-smooth operations.

Let $\mathcal{L}$ be a logic. We say that two graphs G, H (or hypergraphs, or τ-structures) are $(\mathcal{L},)q$-equivalent, and write $G \sim^q_{\mathcal{L}} H$, if G and H satisfy the same $\mathcal{L}$-sentences of quantifier rank q.

We say that $\Box$ is $\mathcal{L}$-smooth if whenever we have

$$G_0 \sim^q_{\mathcal{L}} H_0 \quad \text{and} \quad G_1 \sim^q_{\mathcal{L}} H_1$$

then

$$G_0 \Box G_1 \sim^q_{\mathcal{L}} H_0 \Box H_1.$$

This definition can be adapted to k-ary operations for $k \geq 1$.

Proving that an operation $\Box$ is $\mathcal{L}$-smooth may be difficult. For FOL this can be achieved using Ehrenfeucht-Fraïssé games also know as *pebble games*. Another way of establishing smoothness is via the Feferman-Vaught theorem.

Examples of $\mathcal{L}$-smooth operations:

1. Quantifier-free scalar transductions are both FOL and MSOL-smooth.
2. Quantifier-free vectorized transductions are FOL but not MSOL-smooth.
3. The cartesian product is FOL-smooth but not MSOL-smooth.
   This was shown by A. Mostowski in 1952.
4. The (rich) disjoint union is both FOL and MSOL-smooth.
   The rich disjoint union has two additional unary predicates to distinguish the universes.
   For FOL this was shown by E. Beth in 1952. For MSOL this is due to H. Läuchli, 1966, using Ehrenfeucht-Fraïssé games
5. Adding modular counting quantifiers preserves smoothness.
   For CMSOL and the disjoint union this is due to B. Courcelle, 1990. For CFOL and the product this is due to T. Kotek and J.A. Makowsky, 2012.

3.8 The Finite Rank Theorem

Theorem 3.3 (Godlin, Kotek, Makowsky 2008)**.** *Let f be a numeric parameter or polynomial for τ-structures definable in $\mathcal{L}$ and taking values in an integral domain $\mathcal{R}$. Let $\Box$ be an $\mathcal{L}$-smooth operation. Then the connection matrix $M(f, \Box)$ has finite rank over $\mathcal{R}$.*

The proof uses a Feferman-Vaught-type theorem for graph polynomials, due to B. Courcelle, J.A. Makowsky and U. Rotics, 2000.

Disjoint unions The following graph parameters or not CMSOL-definable because they are maximizing (minimizing) for the disjoint union.

- the clique number $\omega(G)$ and the independence number $\alpha(G)$ of G.
- The chromatic number $\chi(G)$ and the chromatic index $\chi_e(G)$.
- The degrees $\delta(G)$ (minimal), $\Delta(G)$ (maximal)

The same holds for the average degree $d(G)$, but here we use the fact that the Cauchy matrix has growing rank.

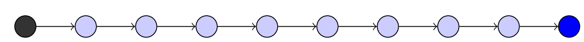
Direct (categorical) products combined with translation schemes. The transduction $\Phi_{sym}(v_1, v_2) = E_D(v_1, v_2) \vee E_D(v_2, v_1)$ transforms a digraph $D = (V_D, E_D)$ into an undirected graph whose edge relation is the symmetric closure of the edge relation of the digraph.

The transduction

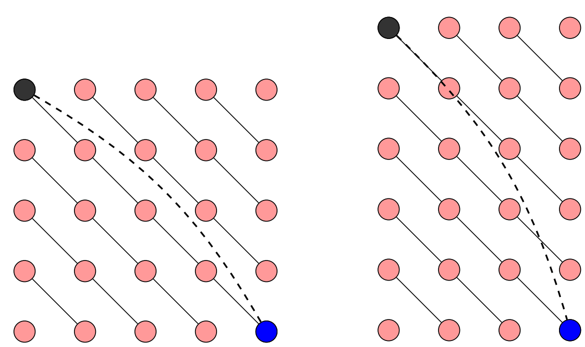
$$\begin{aligned} \Phi_F\left((v_1, v_2), (u_1, u_2)\right) \quad = \quad & (E_1(v_1, u_1) \wedge E_2(v_2, u_2)) \vee \\ & ((v_1, v_2), (u_1, u_2)) = ((start_1, start_2), (end_1, end_2)) \end{aligned}$$

combined with Φ_{sym} transforms the direct product of two directed paths $P^i_{n_i} = (V_1, E_1, start_i, end_i)$ of length n_i with the two constants $start_i$ and end_i, $i = 1, 2$ into an undirected graph with at most one cycle.

When input graphs look like the following,



the result of the transduction is



$n_1 = n_2$ $\qquad$ $n_1 \neq n_2$

Theorem 3.4. *Graphs without cycles of odd (even) length are not* CFOL-*definable even in the presence of a linear order.*

Corollary 3.5. *The following are not definable in* CFOL *with a linear order:*

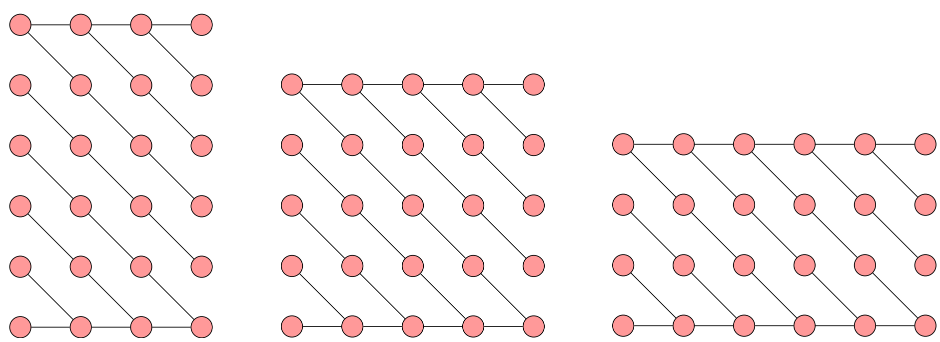
1. *Forests, bipartite graphs, chordal graphs, perfect graphs.*
2. *Interval graphs (cycles are not interval graphs).*
3. *Block graphs (every biconnected component is a clique).*
4. *Parity graphs (any two induced paths joining the same pair of vertices have the same parity).*

Theorem 3.6. *Neither trees nor connected graphs are* CFOL-*definable, even in the presence of a linear order.*

The transduction

$$\begin{aligned}\Phi_T\left((v_1, v_2), (u_1, u_2)\right) \quad = \quad & (E_1(v_1, u_1) \wedge E_2(v_2, u_2)) \vee \\ & (v_1 = u_1 = start_1 \wedge E(v_2, u_2)) \vee \\ & (v_1 = u_1 = end_1 \wedge E(v_2, u_2)),\end{aligned}$$

combined with Φ_{sym} transforms the cartesian product of two directed paths into the structures below:



$n_1 > n_2$ $\qquad$ $n_1 = n_2$ $\qquad$ $n_1 < n_2$

Tree: $n_1 = n_2$. Connected: $n_1 \geq n_2$.

3.8.1 k-graphs and k-sums

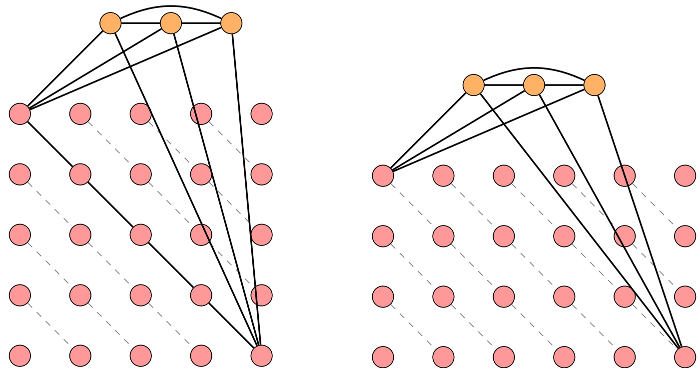
A k-graph is a graph $G = (V(G), E(G))$ with k distinct vertices labeled with $0, 1, \ldots, k-1$. Given two k-graphs G_1, G_2 we define the k-sum $G_1 \sqcup_k G_2$ as the disjoint union of G_1 and G_2 where we identify correspondingly labeled vertices.

Theorem 3.7. *The k-sum is smooth for* $\mathrm{FOL}, \mathrm{CFOL}, \mathrm{MSOL}$ *and* CMSOL.

Theorem 3.8. *Planar graphs are not* CFOL-*definable even on ordered connected graphs*

For our next connection matrix we use the 2-sum of the following two 2-graphs:

- the 2-graph (G, a, b) obtained from from K_5 by choosing two vertices a and b and removing the edge between them.
- the cartesian product of the two graphs $P^1_{n_1}$ and $P^2_{n_2}$:



$n_1 = n_2$ $\qquad\qquad$ $n_1 \neq n_2$

The result of this construction has a clique of size 5 as a minor iff $n_1 = n_2$. It can never have a $K_{3,3}$ as a minor.

If we modify the above construction by taking K_3 instead of K_5 and making $(start_1, start_2)$ and (end_1, end_2) adjacent, we obtain:

Proposition 3.9. *The following classes of graphs are not* CFOL*-definable even on ordered connected graphs.*

1. *Cactus graphs, i.e. graphs in which any two cycles have at most one vertex in common.*
2. *Pseudo-forests, i.e. graphs in which each connected component has at most one cycle.*

3.8.2 Non-definability in CMSOL for graphs $G = (V, E)$: using the join operation

The join operation of graphs $G = (V, E)$, where E is the edge relation, is defined by

$$(V_1, E_1) \bowtie (V_2, E_2) = (V_1 \sqcup V_2, E_1 \sqcup E_2 \cup \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$$

This is a quantifier-free transduction of the disjoint union, hence smooth for CMSOL.

Consider the connection matrix in which the rows and columns are labeled by the graphs E_n consisting of n vertices and no edges.

The graph $E_i \bowtie E_j = K_{i,j}$ is

- hamiltonian iff $i = j$;
- has a perfect matching iff $i = j$;

- is a cage graph (a regular graph with as few vertices as possible for its girth) iff $i = j$;
- is a well-covered graph (every minimal vertex cover has the same size as any other minimal vertex cover) iff $i = j$.

All of these connection matrices have infinite rank.

Proposition 3.10. *None of the properties above are* CMSOL*-definable as graphs even in the presence of a linear order.*

3.8.3 CMSOL **for hypergraph** $G = (V, E; R)$

A hypergraph $G = (V, E; R)$ has vertices V and edges E and an incidence relation R between the two.

- CMSOL for hypergraphs $G = (V, E; R)$ allows quantification over edge sets.
- For the language of hypergraphs the join operation is neither MSOL- nor CMSOL-smooth, since it increases the number of edges.
- Note also that hamiltonicity and having a perfect matching are both definable in CMSOL in the language of hypergraphs.

In the many papers of B. Courcelle, MSOL on graphs is called MSOL_1 and for hypergraphs it is called MSOL_2.

3.8.4 Non-definability in CMSOL for hyper-graphs $G = (V, E; R)$: using the disjoint union

Using the connection submatrices of the disjoint union we still get the properties:

- Regular: $K_i \sqcup K_j$ is regular iff $i = j$;
- A generalization of regular graphs are *bidegree* graphs, i.e., graphs where every vertex has one of two possible degrees. $K_i \sqcup (K_j \sqcup K_1)$ is a bidegree graph iff $i = j$.
- The average degree of $K_i \sqcup E_j$ is at most $\frac{|V|}{2}$ iff $i = j$;
- A digraph is *aperiodic* if the common denominator of the lengths of all cycles in the graph is 1. We denote by C_i^d the directed cycle with i vertices. For prime numbers p, q the digraphs $C_p^d \sqcup C_q^d$ is aperiodic iff $p \neq q$.
- A graph is *asymmetric (or rigid)* if it has no non-trivial automorphisms. It was shown by P. Erdös and A. Rényi (1963) that almost all finite graphs are asymmetric. So there is an infinite set $I \subseteq \mathbb{N}$ such that for $i \in I$ there is an asymmetric graph R_i of cardinality i. $R_i \sqcup R_j$ is asymmetric iff $i \neq j$.

Proposition 3.11. *None of the properties above are* CMSOL*-definable as hypergraphs even in the presence of a linear order.*

3.9 The harmonious chromatic polynomial

Recall that a vertex coloring of a graph G with k colors is *harmonious* if it is proper and each pair of colors appears at most once along an edge. The number of harmonious colorings of G with at most k colors is denoted by $\chi_{harmonious}(G; k)$.

The harmonious index $\chi_{harmonious}(G)$ of a graph G is the smallest k such that there is a harmonious coloring with k colors.

Let iP_2 be the graph which consists of i disjoint edges (in the language of hypergraphs).

Proposition 3.12.
1. $\chi_{harmonious}(iP_2 \sqcup jP_2, k) = 0$ *iff* $i + j > \binom{k}{2}$.
2. $\chi_{harmonious}(iP_2 \sqcup jP_2) = \min_k\{i + j \leq \binom{k}{2}$.
3. $\chi_{harmonious}(G; k)$ *is not* CMSOL-*definable in the language of hypergraphs.*
4. $\chi_{harmonious}(G)$ *is not* CMSOL-*definable in the language of hypergraphs.*

3.9.1 Three graph polynomials

Rainbow polynomial $\chi_{rainbow}(G, k)$ is the number of path-rainbow connected k-colorings, which are functions $c : E(G) \to [k]$ such that between any two vertices $u, v \in V(G)$ there exists a path where all the edges have different colors.

MCC-polynomial For every fixed $t \in \mathbb{N}$, $\chi_{mcc(t)}(G, k)$ is the number of vertex k-colorings $f : V(G) \to [k]$ for which each color class induces a subgraph whose connected components each have size at most t.

Convex coloring polynomial $\chi_{convex}(G, k)$ is the number of convex colorings, i.e., vertex k-colorings $f : V(G) \to [k]$ such that every color induces a connected subgraph of G.

Makowsky and B. Zilber (2005) showed that $\chi_{rainbow}(G, k)$, $\chi_{mcc(t)}(G, k)$, and $\chi_{convex}(G, k)$ are graph polynomials with k as the variable.

Path-rainbow connected colorings were introduced by G. Chartrand et al. in 2008. Their complexity was studied in S. Chakraborty et. al in 2008. $mcc(t)$-colorings were first studied by N. Alon et al. in 2003. Note $\chi_{mcc(1)}(G, k)$ is the chromatic polynomial. Convex colorings were studied by S. Moran in 2007.

Proposition 3.13. *The following connection matrices have infinite rank:*
1. $M(\sqcup_1, \chi_{rainbow}(G, k))$;
2. $M(\sqcup_1, \chi_{convex}(G, k))$;
3. *For every* $t > 0$ *the matrix* $M(\bowtie, \chi_{mcc(t)}(G, k))$.

Proof. $\chi_{rainbow}(G, k)$: We use that the 1-sum of paths with one end labeled is again a path with $P_i \sqcup_1 P_j = P_{i+j-1}$ and that $\chi_{rainbow}(P_r, k) = 0$ iff $r > k + 3$.

$\chi_{convex}(G, k)$: We use edgeless graphs and disjoint union $E_i \sqcup E_j = E_{i+j}$ and that $\chi_{convex}(E_r, k) = 0$ iff $r > k$.

$\chi_{mcc(t)}(G, k)$: We use the join and cliques, $K_i \bowtie K_j = K_{i+j}$ and that $\chi_{mcc(t)}(K_r, k) = 0$ iff $r > kt$. ☐

Corollary 3.14. 1. $\chi_{rainbow}(G, k)$ *and* $\chi_{convex}(G, k)$ *are not* CMSOL-*definable in the language of graphs and hypergraphs.*

2. $\chi_{mcc(t)}(G,k)$ *(for any fixed* $t>0$*) is not* CMSOL*-definable in the language of graphs.*
3. *In particular the chromatic polynomial is not* CMSOL*-definable in the language of graphs.* Note: *It is however* CMSOL*-definable in the language of ordered hypergraphs.*

Proof. (i) The 1-sum and the disjoint union are CMSOL-sum-like[8] and CMSOL-smooth for hypergraphs. (ii) The join is only CMSOL-sum-like and CMSOL-smooth for graphs. □

3.10 Proving non-definability with connection matrices: merits

The advantages of the Finite Rank Theorem for tame $\mathcal{L}$ in proving that a property is not definable in $\mathcal{L}$ are the following:

1. It suffices to prove that certain binary operations on graphs (τ-structures) are $\mathcal{L}$-smooth operations.
2. Once the $\mathcal{L}$-smoothness of a binary operation has been established, proofs of non-definability become surprisingly simple and transparent.
   One of the most striking examples is the fact that asymmetric (rigid) graphs are not definable in CMSOL.
3. Many properties can be proven to be non-definable using the same or similar submatrices, i.e., matrices with the same row and column indices. This has been well illustrated in the examples given above.

3.11 Proving non-definability with connection matrices: limitations

The classical method of proving non-definability in FOL using pebble games is complete in the sense that a property is $\mathrm{FOL}(\tau)_q$-definable iff the class of its models is closed under game equivalence of length q.

Using pebble games one proves easily that the class of structures without any relations of even cardinality, EVEN, is not FOL-definable. However, one cannot prove that EVEN is not FOL-definable using infinite rank connection matrices, in a sense made precise in [16, Proposition 4.1].

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Essential expansion and Property (T)

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We say that a sequence of finite, d-regular graphs is *essentially expander* if it can be turned into a disjoint union of expanders after removing and adding $o(n)$ edges. We give several characterizations of such sequences. We solve Bowen's problem proving that the sofic (Benjamini–Schramm) approximation of a finitely generated group with Kazhdan Property (T) is essentially expander. We use our characterization to reprove a theorem of M. H. Freedman and M. B. Hastings about mapping large subcomplexes of 2-dimensional simplicial complexes to 1-dimensional complexes.

Editors' note: We have added the author's recent preprint [6] to the references.

1 Introduction

Pseudorandomness plays an important role in many mathematical areas. Random-like graphs are well understood in the theory of dense graphs and their limits called graphons [7]. There are many different properties of dense graphs to measure how close is a graph with a given edge density to the Erdős-Rényi random graph $G(n, p)$ with the same edge density: the number of four-cycles (or many other graphs) in the graph, the eigenvalue gap, the maximum difference of the size of a cut and its expected size. The equivalence of these properties (i.e. if one quantity is close to that for the random graph then so are the others) has been proven in a line of work by Thomason, Frankl et. al., Alon and Chung and by Chung, Graham and Wilson, see [1] for an overview. Such (bipartite) graphs are the basic building blocks in Szemerédi's regularity lemma.

Expander graphs are similarly important in the study of sparse graphs, see [9] for the theory of sparse structures, [10] for a short introduction to expanders and [8] for a longer one. Expander graphs have similar characterizations [1] in terms of the eigenvalue gap, in terms of cuts and also in terms of the convergence rate (of the random walk): the latter one is less meaningful in the dense case. However, we cannot expect a local condition in terms of subgraph densities: a random regular graph and a random regular bipartite graph have very different global structures, but both have essentially large girth, i.e. very similar local statistics. Hence we can at most hope for the **forceability of expansion**, a necessary condition in terms of the local statistics that implies an expander-like structure. But even such hopes seem to be beyond reach: we cannot distinguish between a graph and its two disjoint copies. The latter is a disconnected graph with bad expansion. We introduce the notion of **essential expansion** to handle this phenomenon.

2 Main results

A sequence of d-regular graphs $\{G_n\}_{n=1}^{\infty}$ is an *expander* sequence if there exists $\delta > 0$ such that for all n, $S \subseteq V(G_n)$, where $|S| \leq |V(G_n)|/2$, the number of edges leaving S is at least $\delta|S|$.

Definition 2.1. A sequence of d-regular graphs $\{G_n\}_{n=1}^{\infty}$ is *essentially expander* if there exists a $\delta > 0$ such that for every n the graph G_n is the vertex-disjoint union of δ-expanders modulo $o(n)$ edges.

The next theorem [5] (conjectured by Bowen in [3]) shows that essential expansion is forceable.

Theorem 2.2. *If a sequence of d-regular graphs $\{G_n\}_{n=1}^{\infty}$ Benjamini-Schramm converges to the Cayley graph of a finitely generated group with Kazhdan Property (T) then it is essentially expander.*

This rhymes with the Connes-Weiss definition of **Kazhdan Property (T)**: a group has Property (T) iff every ergodic action of the group on a probability measure space is expander. There are examples of finitely presented groups with Property (T), hence there are easy descriptions of a local statistics that force essential expansion. See [2] for more on groups with Property (T).

We give a spectral characterization of essentially expander graph sequences.

Theorem 2.3. *Consider a sequence of d-regular graphs $\{G_n\}_{n=1}^{\infty}$. The following are equivalent:*

- *There exists an $\varepsilon > 0$ such that for every $k > 0$ the following holds: for every $n > 0$ and $S \subseteq V(G_n)$ we have the inequality $\|A^{k+1}\chi_S - A^k\chi_S\| \leq (1-\varepsilon)^k\|A\chi_S - \chi_S\| + o_n(1)$.*
- *The sequence is essentially expander.*

Note that the first condition is easy to check for an essentially expander sequence. Theorem 2.2 is a straightforward consequence of Theorem 2.3.

3 Applications

Ergodic decomposition theorems usually require separability conditions. Theorem 2.2 gives an ergodic decomposition theorem **beyond the separable universe**: it implies that an ergodic decomposition exists for limits (ultraproduct, weak limit etc.) of a sequence of graphs Benjamini-Schramm converging to the Cayley graph of a finitely generated group with Property (T). This was a conjecture of Bowen [3]. The theorem can be generalized to limits of graph sequences. This makes it possible to prove an ergodic decomposition theorem for sequences of essentially strongly ergodic graphs. Here the components will be strongly ergodic. The key tool is a generalization of Theorem 2.3 to essentially strongly ergodic graphs, see [5] for details.

We call a sequence of graphs $\{G_n\}_{n=1}^{\infty}$ **hyperfinite** if for every $\varepsilon > 0$ there exists $K > 0$ such that for every n we can delete εn edges of $V(G_n)$ and get a graph that has connected components of size at most K. Planar graphs are good examples of hyperfinite graphs. Graphs with large girth and average degree greater than $(2+\varepsilon)$ are not hyperfinite. The group-theoretical analogue of hyperfiniteness is amenability: every sequence of graphs that Benjamini-Schramm converges to the Cayley graph of a finitely generated amenable group is hyperfinite. Hyperfinite graphs play an important role in property testing.

Freedman and Hastings introduced a higher dimensional analogue and defined **hyperfinite simplicial complexes**. We call a sequence of k-dimensional simplicial complexes $\{X_n\}_{n=1}^{\infty}$, with 0-skeletons $sk_0(X_n)$, *k-hyperfinite* if for every $\varepsilon > 0$ there is a D such that for every n there is an X'_n obtained by the removal of $\varepsilon|sk_0(X_n)|$ k-dimensional simplices from X_n, a $(k-1)$-dimensional simplicial complex Y_n, and a continuous function $f_n : X'_n \to Y_n$, such that the pre-image of every element of Y_n has diameter at most D. The structure of such complexes can be understood via reductions to complexes of smaller dimension. Freedman and Hastings constructed non-hyperfinite complexes as part of an attempt to attack the **quantum PCP conjecture**. They generalized the hypergraph product codes of Tillich and Zémor. We give new constructions and a simple proof in the 2-dimensional case, which is sufficient for the above applications.

Theorem 3.1. *Consider a finitely presented sofic group Γ with Property (T) and a sequence of finite 2-simplices $\{X_n\}_{n=1}^{\infty}$ that Benjamini-Schramm converges to the Cayley complex of Γ. Then $\{X_n\}_{n=1}^{\infty}$ is not 2-hyperfinite.*

We give a sketch of the idea of the proof. We proceed by supposing for a contradiction that there are X'_n, Y_n, f_n establishing 2-hyperfiniteness for every n. Now each Y_n is a 1-dimensional simplex and its covers can have an almost arbitrary global structure, since its fundamental group is a free group. There exists one far from an essential expander: this can be checked using the spectral condition of Theorem 2.3. On the other hand, the structure of this cover can be pulled back to a cover of X'_n, which is close to X_n in the Benjamini-Schramm sense. Hence the 1-skeleton of this cover should be close to an essential expander due to Theorem 2.2, and this gives the desired contradiction.

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Algebraic and model-theoretic methods in constraint satisfaction

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This text is related to the topic which I presented in the Doc-course at Charles University Prague in the fall of 2014. It should be considered as a supplement to the course rather than a summary: for the present text I chose a different focus, developing the subject around the notion of a function clone instead of starting with constraint satisfaction problems. Moreover, I concentrate here on possible future research in the field more than on the details of what is already known; the latter can be found in the literature listed in the references at the end.

1 Overview

A *function clone* is a set $\mathscr{C}$ of finitary functions on a set D which is closed under composition and which contains all projections. More formally,

- whenever $f \in \mathscr{C}$ is n-ary, and $g_1, \ldots, g_n \in \mathscr{C}$ are m-ary, then the m-ary function $f(g_1, \ldots, g_n)$ defined by

  $$(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

  is an element of $\mathscr{C}$;

- for all $1 \leq k \leq n < \omega$, $\mathscr{C}$ contains the *k-th n-ary projection* $\pi^n_k \colon D^n \to D$, uniquely defined by the equation $\pi^n_k(x_1, \ldots, x_n) = x_k$.

There are two main sources of function clones:

- the term operations of any algebra $\mathfrak{A}$ form a clone, the *term clone* of $\mathfrak{A}$ (and in fact, every clone is of this form);

- the set of all operations which *preserve* a given relational structure Γ form a clone, called the *polymorphism clone* of Γ (certain clones, the *topologically closed clones*, are of this form).

The first source of function clones makes them an object of primary interest in universal algebra, since many properties of an algebra, such as its subalgebras and congruences, only depend on its term operations. The second source links them with relational structures, and in particular, as we will see, with certain questions in complexity theory.

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This topic of this text is function clones over a countably infinite set and their applications in complexity theory. While the investigation of *all* such clones is not very promising since in the general setting, hardly any positive structural results could be expected (cf. for example [26]), research on clones which are "sufficiently rich" has proven extremely fruitful in recent years [11, 12, 5, 36]. We are interested here in function clones which are rich in the sense that they contain a rather large permutation group: a permutation group on a countably infinite set D is called *oligomorphic* iff its componentwise action on D^n has only finitely many orbits, for all $n \geq 1$ (cf. [23]). Function clones containing an oligomorphic permutation group, referred to as *oligomorphic clones* [5], have been shown to enjoy many properties of function clones on finite sets. For example, they satisfy a topological variant of Birkhoff's HSP theorem; moreover, they encode the complexity of certain computational problems, so-called *constraint satisfaction problems (CSPs)*, and indeed have proven to be a valuable tool in the study of the complexity of such problems in what is called the *algebraic approach* to CSPs. Oligomorphic function clones encode a much larger class of CSPs than function clones over finite sets [4], and yet many tools from the finite carry over to the oligomorphic setting.

Of particular importance to us will be oligomorphic clones which arise from *homogeneous* relational structures in a finite language, and some of our methods rely on *Ramsey theory* and on connections of Ramsey-type theorems with topological dynamics. While the original motivation for studying function clones comes from universal algebra, and later and independently from constraint satisfaction problems, their study therefore also involves tools and concepts from model theory, combinatorics, and topological dynamics.

2 The state of the art

Birkhoff's theorem for oligomorphic clones. I will start by recalling the finite version of Birkhoff's HSP theorem. An *algebra* is a structure with a purely functional signature. The *clone of an algebra* $\mathfrak{A}$ with signature τ, denoted by $\mathrm{Clo}(\mathfrak{A})$, is the set of all functions with finite arity on the domain A of $\mathfrak{A}$ which can be written as τ-terms over $\mathfrak{A}$. More precisely, every abstract τ-term t naturally induces a finitary function $t^{\mathfrak{A}}$ on A, and $\mathrm{Clo}(\mathfrak{A})$ consists precisely of the functions of this form.

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras of the same signature τ. The assignment ξ from $\mathrm{Clo}(\mathfrak{A})$ to $\mathrm{Clo}(\mathfrak{B})$ which sends every element $t^{\mathfrak{A}}$ of $\mathrm{Clo}(\mathfrak{A})$ to $t^{\mathfrak{B}}$ is a well-defined function if and only if for all τ-terms s, t we have that $s^{\mathfrak{B}} = t^{\mathfrak{B}}$ whenever $s^{\mathfrak{A}} = t^{\mathfrak{A}}$. In that case, ξ is in fact a surjective *clone homomorphism*, and we then call ξ the *natural homomorphism* from $\mathrm{Clo}(\mathfrak{A})$ onto $\mathrm{Clo}(\mathfrak{B})$. In general, a *clone homomorphism* is a function $\sigma\colon \mathscr{C} \to \mathscr{D}$, where $\mathscr{C}, \mathscr{D}$ are clones (possibly acting on different base sets), which sends functions in $\mathscr{C}$ to functions of the same arity in $\mathscr{D}$, every projection in $\mathscr{C}$ to the corresponding projection in $\mathscr{D}$, and which preserves composition, i.e., $\sigma(f(g_1, \ldots, g_n)) = \sigma(f)(\sigma(g_1), \ldots, \sigma(g_n))$ for all n-ary $f \in \mathscr{C}$ and all m-ary $g_1, \ldots, g_n \in \mathscr{D}$, for all $n, m \geq 1$ (cf. [16]).

When $\mathcal{C}$ is a class of algebras with common signature τ, then $\mathrm{P}(\mathcal{C})$ denotes the class of all products of algebras from $\mathcal{C}$, $\mathrm{P}^{\mathrm{fin}}(\mathcal{C})$ denotes the class of all *finite* products of algebras from $\mathcal{C}$, $\mathrm{S}(\mathcal{C})$ denotes the class of all subalgebras of algebras from $\mathcal{C}$, and $\mathrm{H}(\mathcal{C})$ denotes the class of all homomorphic images of algebras from $\mathcal{C}$ (when defining these operators, we consider algebras up to isomorphism). A *pseudovariety* is a class $\mathcal{V}$ of algebras of the same signature such that $\mathcal{V} = \mathrm{H}(\mathcal{V}) = \mathrm{S}(\mathcal{V}) = \mathrm{P}^{\mathrm{fin}}(\mathcal{V})$, i.e., a class closed under homomorphic images, subalgebras, and finite products; the pseudovariety *generated* by a class of algebras $\mathcal{C}$ (or by a single algebra $\mathfrak{A}$) is the smallest pseudovariety that contains $\mathcal{C}$ (contains $\mathfrak{A}$, respectively). For *finite* algebras, Birkhoff's HSP theorem takes the following form (see Exercise 11.5 in combination with the proof of Lemma 11.8 in [22]).

Theorem 1 (Birkhoff [3])**.** *Let $\mathfrak{A}, \mathfrak{B}$ be finite algebras with the same signature. Then the following three statements are equivalent.*

1. *The natural homomorphism from* $\mathrm{Clo}(\mathfrak{A})$ *onto* $\mathrm{Clo}(\mathfrak{B})$ *exists.*
2. $\mathfrak{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A})$*.*
3. $\mathfrak{B}$ *is contained in the pseudovariety generated by* $\mathfrak{A}$*.*

When $\mathfrak{A}$ and $\mathfrak{B}$ are of arbitrary cardinality, then the equivalence of (2) and (3) still holds; however, if one wants to maintain equivalence with item (1), then another version of Birkhoff's theorem states that one has to replace finite powers by arbitrary powers in the second item, that is, one has to replace $\mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A})$ by $\mathrm{HSP}(\mathfrak{A})$; the third item has to be adapted using the notion of a *variety* of algebras, i.e., a class of algebras of common signature closed under the operators H, S and P.

It recently turned out that one can prove a similar theorem with finite powers for algebras $\mathfrak{A}$ on a countably infinite domain whose clone $\mathrm{Clo}(\mathfrak{A})$ is oligomorphic – we call such algebras *oligomorphic* as well. To this end, one has to see function clones not only as algebraic, but also as topological objects. On any set D, there is a largest function clone $\mathscr{O}_D$: the clone of all finitary operations on D. The "function space" $\mathscr{O}_D$ carries a natural topology, namely the topology of pointwise convergence, with respect to which the composition of functions is continuous. A basis of open sets of this topology is given by the sets of the form

$$\{f \colon D^k \to D \mid f(a_1^1, \dots, a_k^1) = a_0^1, \dots, f(a_1^n, \dots, a_k^n) = a_0^n\}\ .$$

In fact, similarly to the Baire space $\mathbb{N}^{\mathbb{N}}$, $\mathscr{O}_D$ then becomes a Polish space (cf. for example [16]). As a subset of $\mathscr{O}_D$, every function clone $\mathscr{C}$ on D inherits this topology, and hence carries a topological structure in addition to its algebraic structure given by the equations which hold in $\mathscr{C}$. We denote the topological closure of a function clone $\mathscr{C}$ in $\mathscr{O}_D$ by $\overline{\mathscr{C}}$.

It is not hard to see that all algebras in the pseudovariety generated by an oligomorphic algebra are again oligomorphic. The following is the topological variant of Birkhoff's theorem for oligomorphic algebras.

Theorem 2 (Bodirsky and Pinsker [11])**.** *Let $\mathfrak{A}, \mathfrak{B}$ be oligomorphic or finite algebras with the same signature. Then the following three statements are equivalent.*

1. *The natural homomorphism from $\overline{\mathrm{Clo}(\mathfrak{A})}$ onto $\overline{\mathrm{Clo}(\mathfrak{B})}$ exists and is continuous.*
2. $\mathfrak{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A})$.
3. $\mathfrak{B}$ *is contained in the pseudovariety generated by* $\mathfrak{A}$.

Note that Theorem 1 really is a special case of Theorem 2, since the topology of any function clone on a finite set is discrete, and hence the natural homomorphism from the clone of a finite algebra to that of another algebra is always continuous.

Applications to constraint satisfaction problems. Let us now turn to applications of oligomorphic function clones to computational complexity problems. Every relational structure Γ in a finite language defines a computational problem, called the *constraint satisfaction problem* of Γ and denoted by $\mathrm{CSP}(\Gamma)$, as follows: input of the problem is a primitive positive sentence ϕ in the language for Γ, i.e., a sentence of the form $\exists x_1, \ldots, x_n(\phi_1 \wedge \cdots \wedge \phi_m)$ where $\phi_1, \ldots, \phi_m$ are atomic formulas; the problem is to decide whether or not ϕ holds in Γ. An instance of this problem therefore asks about the existence of elements of Γ satisfying a given conjunction of atomic conditions. The structure Γ is called the *template* of the problem, and can be finite or infinite. We will later see how infinite templates can model natural computational problems, and refer also to [4] for an abundance of examples. We remark that $\mathrm{CSP}(\Gamma)$ is often presented in the form of a *homomorphism problem*, which is easily seen to be equivalent: in this formulation, the input is a *finite* structure Ω in the language of Γ (which can still be finite or infinite), and the question is whether or not there exists a homomorphism from Ω into Γ.

To every relational structure Γ, one can assign a function clone on the domain of Γ as follows. A *polymorphism* of a structure Γ is a homomorphism from Γ^k to Γ for some finite $k \geq 1$; the *polymorphism clone* $\mathrm{Pol}(\Gamma)$ of Γ is the set of all polymorphisms of Γ. It is easy to see that $\mathrm{Pol}(\Gamma)$ is a function clone which is closed in the pointwise convergence topology described above, and in fact, the closed function clones are precisely the function clones of the form $\mathrm{Pol}(\Gamma)$ for a relational structure Γ.

For finite relational structures Γ, the complexity of $\mathrm{CSP}(\Gamma)$ depends, up to polynomial time, only on $\mathrm{Pol}(\Gamma)$ (cf. [21, 20]); this fact is the basis of the approach to constraint satisfaction via clones. The same is true for structures with oligomorphic polymorphism clones [10]. But which structures have oligomorphic polymorphism clones? The answer can be found in a classical theorem of model theory, the theorem of Engeler, Svenonius, and Ryll-Nardzewski (see e.g. the textbook [28]). A countable structure Γ is called *ω-categorical* iff all countable models of the first-order theory of Γ are isomorphic to Γ. Now the theorem states that the automorphism group $\mathrm{Aut}(\Gamma)$ of a countable structure Γ is oligomorphic if and only if Γ is ω-categorical. It follows that the polymorphism clone of a countable structure Γ is oligomorphic if and only if Γ is ω-categorical.

Theorem 3 (Bodirsky and Nešetřil [10]). *Let Γ, Γ' be ω-categorical structures in finite relational languages which have the same domain. If $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Gamma')$, then $\mathrm{CSP}(\Gamma')$ is polynomial-time reducible to $\mathrm{CSP}(\Gamma)$.*

As a consequence, for ω-categorical structures Γ the complexity of their CSP is still up to polynomial time encoded in their polymorphism clone, i.e., if $\mathrm{Pol}(\Gamma') = \mathrm{Pol}(\Gamma)$, then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(\Gamma')$ are polynomial-time equivalent.

The theory of the algebraic approach to CSPs goes much further, which brings us back to Birkhoff's HSP theorem. For a structure Γ, we call any algebra on the domain on Γ whose functions are precisely the elements of $\mathrm{Pol}(\Gamma)$ indexed in some arbitrary way a *polymorphism algebra* of Γ. It can be shown that if Γ and Γ' are finite structures in a finite relational language, and a polymorphism algebra $\mathfrak{B}$ of Γ' is contained in the pseudovariety of a polymorphism algebra $\mathfrak{A}$ of Γ, then $\mathrm{CSP}(\Gamma')$ is polynomial-time reducible to $\mathrm{CSP}(\Gamma)$ [21, 20]. By Birkhoff's theorem, this is the case iff the natural homomorphism from $\mathrm{Clo}(\mathfrak{A})$ onto $\mathrm{Clo}(\mathfrak{B})$ exists. One then sees that this is the case if and only if there exists a surjective clone homomorphism from $\mathrm{Pol}(\Gamma)$ onto $\mathrm{Pol}(\Gamma')$. Hence, the complexity of the CSP of a finite relational structure Γ only depends on the abstract structure of the clone $\mathrm{Pol}(\Gamma)$. Similarly to abstract groups, *abstract clones* can be formalized as multi-sorted algebras equipped with composition operations as well as with constant symbols for the projections (cf. , for example, [26] or [16]), but one can avoid this technicality: in practice, it is enough to know that abstract clones simply encode the equations which hold between its functions, or more precisely, it is enough to know that clone homomorphisms as defined above are precisely the structure preserving maps between those objects.

Using the topological generalization of Birkhoff's theorem, one can show the following for ω-categorical structures (this is a simplified version; for a stronger formulation see [11]).

Theorem 4 (Bodirsky and Pinsker [11]). *Let Γ, Γ' be finite or ω-categorical structures in a finite relational language. If there exists a surjective continuous clone homomorphism from $\mathrm{Pol}(\Gamma)$ onto $\mathrm{Pol}(\Gamma')$, then $\mathrm{CSP}(\Gamma')$ is polynomial-time reducible to $\mathrm{CSP}(\Gamma)$.*

In particular, for ω-categorical structures the complexity of their CSP is still up to polynomial time encoded in their polymorphism clone, seen as an abstract clone together with the topology on the functions. In analogy to topological groups, we call such objects *topological clones* [16]. More precisely, if two ω-categorical structures Γ, Γ' have polymorphism clones which are isomorphic as topological clones (i.e., via a bijection which is a clone homomorphism, whose inverse is a clone homomorphism, and which is a homeomorphism), then their CSPs are polynomial-time equivalent.

A class of ω-categorical structures for which the CSP is of particular interest are structures with a first-order definition in a homogeneous structure in a finite language. A structure Δ is called *homogeneous* iff any isomorphism between finitely generated substructures of Δ extends to an automorphism of Δ (some authors call this notion *ultrahomogeneity* to distinguish it from related concepts of homogeneity). A *reduct* of a structure Δ is a relational structure on the same domain each of whose relations can be defined in Δ by a first-order formula without parameters. Countable homogeneous structures in a finite relational language are ω-categorical, and reducts of ω-categorical structures are ω-categorical as well, and hence fall into our context (cf. the textbook [28]).

When Γ is the reduct of a homogeneous structure in a finite language, then CSP(Γ) models a certain type of problem about finitely generated structures, as we will outline in the following. Homogeneous structures can be seen as generic objects, called *Fraïssé limits*, representing so-called *Fraïssé classes* of finitely generated structures. A *Fraïssé class* is a class $\mathcal{C}$ of finitely generated structures in a fixed countable language closed under isomorphism and induced substructures which satisfies the joint embedding property, i.e., for all $\Omega_0, \Omega_1 \in \mathcal{C}$ there is $\Omega_2 \in \mathcal{C}$ such that Ω_0, Ω_1 embed into Ω_2, and the *amalgamation property*, i.e., for any three structures $\Omega_0, \Omega_1, \Omega_2$ in $\mathcal{C}$ and embeddings $e : \Omega_0 \to \Omega_1$ and $f : \Omega_0 \to \Omega_2$ there exists $\Omega_3 \in \mathcal{C}$ and embeddings $e' : \Omega_1 \to \Omega_3$ and $f' : \Omega_2 \to \Omega_3$ such that $e' \circ e = f' \circ f$. For any Fraïssé class $\mathcal{C}$ there exists an up to isomorphism unique homogeneous structure $\Delta_{\mathcal{C}}$, called the Fraïssé limit of $\mathcal{C}$, whose *age*, i.e., the class of its finitely generated substructures up to isomorphism, equals $\mathcal{C}$. Conversely, the age of any homogeneous structure in a countable language is a Fraïssé class.

For example, the *random graph* $G = (V, E)$ is the Fraïssé limit of the class of finite undirected graphs without loops, and similarly there exist a *random partial order*, a *random tournament*, *random hypergraphs*, a *random digraph*, and so forth. Let us stick to the first example for a moment and let us define a class of computational problems about finite graphs as follows. Call quantifier-free formulas in the language of graphs *graph formulas*. A graph formula $\Phi(x_1, \ldots, x_m)$ is *satisfiable in a graph* iff there exists a graph H and an m-tuple a of elements in H such that $\Phi(a)$ holds in H. Now let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a finite set of graph formulas. Then Ψ gives rise to the following computational problem.

Graph-SAT(Ψ)
INSTANCE: A set of variables W and a graph formula of the form $\Phi = \phi_1 \wedge \cdots \wedge \phi_l$ where each ϕ_i for $1 \leq i \leq l$ is obtained from one of the formulas ψ in Ψ by substituting the variables from ψ by variables from W.
QUESTION: Is Φ satisfiable in a graph?

In words, an instance of Graph-SAT(Ψ) asks whether there exists a (finite) graph with satisfies a conjunction of properties; which properties can appear is restricted by the fixed set of graph formulas Ψ. Therefore, the computational complexity increases with Ψ in the sense that $\Psi \subseteq \Psi'$, then any algorithm for Graph-SAT(Ψ') solves Graph-SAT(Ψ). It is easy to see that each problem

Graph-SAT(Ψ) is in NP, i.e., solvable in nondeterministic polynomial time.

The connection with CSPs is that every problem Graph-SAT(Ψ) can be translated into CSP(Γ_Ψ) for a finite language reduct Γ_Ψ of the random graph $G = (V, E)$ and vice-versa. For one direction, let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a set of graph formulas. To this set we assign a reduct Γ_Ψ of G which has for each ψ_i a relation R_i consisting of those tuples of elements of V that satisfy ψ_i (where the arity of R_i is given by the number of distinct variables that occur in ψ_i). One readily sees that any algorithm for Graph-SAT(Ψ) can be adapted to CSP(Γ_Ψ) and vice-versa, and so the problems are essentially the same. For the other direction, if Γ is a reduct of G in a finite language, then each of its relations is defined by a first-order formula over G, and indeed even by a quantifier-free first-order formula (i.e., a graph formula), since homogeneity and ω-categoricity imply quantifier elimination. Let Ψ_Γ the set of those graph formulas. Again, one easily checks that Graph-SAT(Ψ_Γ) and CSP(Γ) are basically the same problem [13].

Now let $\mathcal{C}$ be an arbitrary Fraïssé class of finitely generated structures in a finite language. As in the case of graphs, for a finite set Ψ of quantifier-free first-order formulas in the language of $\mathcal{C}$, we can define the following computational problem.

$\mathcal{C}$-SAT(Ψ)
INSTANCE: A set of variables W and a formula of the form $\Phi = \phi_1 \wedge \cdots \wedge \phi_l$ where each ϕ_i for $1 \leq i \leq l$ is obtained from one of the formulas ψ in Ψ by substituting the variables from ψ by variables from W.
QUESTION: Is Φ satisfiable in a structure in $\mathcal{C}$?

As before, each problem $\mathcal{C}$-SAT(Ψ) is equivalent to CSP(Γ_Ψ) for an appropriate reduct Γ_Ψ of the Fraïssé limit $\Delta_\mathcal{C}$ of $\mathcal{C}$ and vice-versa. Hence, classifying the complexity of the problems $\mathcal{C}$-SAT(Ψ) and classifying the complexity of the constraint satisfaction problems of reducts of $\Delta_\mathcal{C}$ is one and the same thing. Complete classifications have been obtained so far for the following countable homogeneous structures.

- the empty structure $(\mathbb{N}, =)$ [7];
- the order of the rationals $(\mathbb{Q}, \leq)$ [9];
- the random graph $G = (V, E)$ [13].

In each of the three cases the classifications resulted in dichotomies: the CSPs of the reducts turned out to be either NP-complete or in P (i.e., solvable in polynomial time, which we will henceforth refer to as *tractable*). While there exist CSPs of homogeneous digraphs which are undecidable [18], the following representability condition for a Fraïssé class $\mathcal{C}$, arguably reasonable for the most interesting computational problems, forces $\mathcal{C}$-SAT problems to be in NP, and could possibly imply a general dichotomy. Let τ be a finite relational signature. A class $\mathcal{C}$ of finite τ-structures is called *finitely bounded* iff there exists a finite set of finite τ-structures $\mathscr{F}$ such that $\mathcal{C}$ consists precisely of those finite τ-structures which do not embed any element of $\mathscr{F}$. A relational structure is called finitely bounded iff its age is finitely bounded. When Γ is a finite language reduct of a finitely bounded homogeneous structure Δ, then $\mathrm{CSP}(\Gamma)$ is easily seen to be in NP. We conjecture the following.

Conjecture 5. *Let Δ be a finitely bounded homogeneous structure, and let Γ be a finite language reduct of Δ. Then $\mathrm{CSP}(\Gamma)$ is either in P or NP-complete.*

The advantage of translating $\mathcal{C}$-SAT problems into constraint satisfaction problems of reducts of the Fraïssé limit $\Delta_{\mathcal{C}}$ is that it allows for the algebraic approach via clones, as we will now outline.

Firstly, by Theorem 3 we have that if Γ, Γ' are ω-categorical structures on the same domain and $\mathrm{Pol}(\Gamma) \supseteq \mathrm{Pol}(\Gamma')$, then $\mathrm{CSP}(\Gamma)$ has a polynomial-time reduction to $\mathrm{CSP}(\Gamma')$. On a theoretical level, this implies that when one wants to classify the complexity of the CSPs of all reducts of $\Delta_{\mathcal{C}}$, it suffices to consider all polymorphism clones of reducts of $\Delta_{\mathcal{C}}$ – reducts with equal polymorphism clones are polynomial-time equivalent. Those clones are precisely the closed function clones which contain the automorphism group $\mathrm{Aut}(\Delta_{\mathcal{C}})$ of $\Delta_{\mathcal{C}}$. Moreover, if a closed function clone corresponds to a tractable (i.e., polynomial-time solvable) CSP, then so do all closed function clones containing it; if it corresponds to a NP-hard CSP, then so do all closed function clones above $\mathrm{Aut}(\Delta_{\mathcal{C}})$ contained in it. In the case of an existing dichotomy, one thus has to find the border between tractability and NP-hardness in the lattice of closed function clones containing $\mathrm{Aut}(\Delta_{\mathcal{C}})$. On a practical level, this implies that if the CSP of a reduct Γ is in P, then this is witnessed by the presence of certain functions in $\mathrm{Pol}(\Gamma)$. And indeed, the presence of polymorphisms with certain properties have been successfully translated into algorithms in the classifications above – see [12].

The second use of polymorphism clones is that Theorem 4 allows us to compare CSPs on different domains, resulting in a tool both for showing tractability as well as for showing hardness. As for the latter, it turns out to be convenient to show NP-hardness of $\mathrm{CSP}(\Gamma)$ by exposing a continuous clone homomorphism from $\mathrm{Pol}(\Gamma)$ onto $\mathrm{Pol}(\Gamma')$, where Γ' is an ω-categorical or finite structure with a hard CSP [11]. In practice, Γ' will generally be finite; in fact, Γ' will often be any structure on a two-element set with a *trivial* polymorphism clone, i.e., the clone $\mathbf{1}$ of all projections on a two-element set (which is the polymorphism clone of NP-complete structures). Since $\mathbf{1}$ is isomorphic to the smallest function clone (i.e., the clone of projections) on any finite set with at least two elements, and on finite domain smaller polymorphism clones correspond to harder CSPs, a continuous clone homomorphism to $\mathbf{1}$ is in a sense the strongest finite reason for NP-hardness (more precisely, it implies that Γ *pp-interprets* all finite structures [11]; see also [17]).

Notation 6. We write $\mathbf{1}$ for the clone of all projections on a two-element set.

It is an open conjecture, and indeed the main conjecture for CSPs of finite structures known as the *tractability conjecture*, that under a cosmetic assumption (the assumption of having an *idempotent* polymorphism clone, see Section 3) on a finite structure Γ, the CSP for Γ is NP-complete if there exists a clone homomorphism from Pol(Γ) onto **1**, and in P otherwise. Clearly, if there exists such a homomorphism, then CSP(Γ) is NP-complete; the open part is the other direction. Note that if there is no such homomorphism, then this is witnessed by equations which hold in Pol(Γ) but which cannot be satisfied in **1**. Numerous equations have been translated into algorithms, and indeed every non-trivial set of equations of an idempotent clone translates into an algorithm if one believes in the tractability conjecture. In the ω-categorical setting, we cannot purely rely on equations, but need to take into account the topology on the functions – at least with what we know today. Recently, research has been conducted investigating the role of this topology [17], and about how to show tractability in case there exists no continuous homomorphism to **1**.

A Ramsey-theoretic method. The behavior of polymorphism clones of reducts of homogeneous structures Δ in a finite relational language seems to be particularly close to that of function clones on finite sets when Δ satisfies a particular combinatorial property. Let τ be a relational signature. We say that a class $\mathcal{C}$ of finite τ-structures is a *Ramsey class* (in the sense of [37]) iff for all $\Omega_0, \Omega_1 \in \mathcal{C}$ there exists $\Omega_2 \in \mathcal{C}$ such that for all colorings of the copies of Ω_0 in Ω_2 with two colors there exists an isomorphic copy Ω_1' of Ω_1 in Ω_2 such that all copies of Ω_0 in Ω_1' have the same color. A relational structure is called *Ramsey* iff its age is a Ramsey class. When $\mathcal{C}$ is a relational Fraïssé class and $\Delta_{\mathcal{C}}$ its Fraïssé limit, then it is equivalent to call $\Delta_{\mathcal{C}}$ Ramsey iff for all $\Omega_0, \Omega_1 \in \mathcal{C}$ and all colorings of the copies of Ω_0 in $\Delta_{\mathcal{C}}$ there exists a copy of Ω_1 in $\Delta_{\mathcal{C}}$ on which the coloring is constant. Examples of Fraïssé classes which are Ramsey classes are the class of finite ordered undirected graphs, the class of finite linear orders, and the class of finite partial orders with a linear extension [39, 38, 1].

We remark that, for example, neither the random graph nor the random partial order are Ramsey, and thus seem to fall out of this framework. However, they are themselves reducts of homogeneous Ramsey structures, namely the random ordered graph (i.e., the Fraïssé limit of the class of finite ordered undirected graphs) and the random partial order with a random linear extension (i.e., the Fraïssé limit of the class of finite partial orders with a linear extension).

The Ramsey property can be exploited as follows. Let Ξ be a structure. The *type* of a tuple $b = (b^1, \ldots, b^n)$ of elements of Ξ, denoted by $\mathrm{tp}(b)$, is the set of first-order formulas $\phi(x_1, \ldots, x_n)$ such that $\phi(b^1, \ldots, b^n)$ holds in Ξ. Now let $\Xi_1, \ldots, \Xi_m$ be structures. For an element a of the product $\Xi_1 \times \cdots \times \Xi_m$ and $1 \leq i \leq m$, we write a_i for the i-th coordinate of a. The *type* of a tuple $(a^1, \ldots, a^n)$ of elements $a^1, \ldots, a^n \in \Xi_1 \times \cdots \times \Xi_m$, denoted by $\mathrm{tp}(a^1, \ldots, a^n)$, is the m-tuple containing the types of $(a_i^1, \ldots, a_i^n)$ in Ξ_i for each $1 \leq i \leq m$. A function $f : \Xi_1 \times \cdots \times \Xi_m \to \Omega$ is called *canonical* iff it sends finite tuples of equal type in $\Xi_1 \times \cdots \times \Xi_m$ to tuples of equal type in Ω; that is, whenever $\mathrm{tp}(a^1, \ldots, a^n) = \mathrm{tp}(b^1, \ldots, b^n)$, then $\mathrm{tp}(f(a^1), \ldots, f(a^n)) = \mathrm{tp}(f(b^1), \ldots, f(b^n))$. For a relational structure Δ and elements $c_1, \ldots, c_n$ of Δ, we write $(\Delta, c_1, \ldots, c_n)$ for the expansion of Δ by the constants $c_1, \ldots, c_n$. The structure Δ is *ordered* iff it has a linear order among its relations. Now the following holds [18, 12].

Theorem 7 (Bodirsky, Pinsker and Tsankov [18]). *Let Δ be an ordered homogeneous Ramsey structure in a finite relational language, and let $\mathscr{C} \supseteq \mathrm{Aut}(\Delta)$ be a closed function clone. Then for all $f \in \mathscr{C}$ and all $c_1, \ldots, c_n \in \Delta$ there exists a function $g \in \mathscr{C}$ which is canonical as a function on $(\Delta, c_1, \ldots, c_n)$, and which agrees with f on $\{c_1, \ldots, c_n\}$.*

Thus under these conditions on Δ, if there is a function f in a polymorphism clone of a reduct of Δ which does something of interest (e.g., algorithmically) on a finite set $\{c_1, \ldots, c_n\}$, then there is also a canonical function in this clone which does the same. Note that canonical functions on $(\Delta, c_1, \ldots, c_n)$ are finite objects in the following sense. Every canonical function $f : (\Delta, c_1, \ldots, c_n)^m \to (\Delta, c_1, \ldots, c_n)$ defines an m-ary function $T(f)$ on the types of $(\Delta, c_1, \ldots, c_n)$ in an obvious way, by the very definition of canonicity. Moreover, this type function $T(f)$ determines f in the sense that if two canonical functions $f, g : (\Delta, c_1, \ldots, c_n)^m \to (\Delta, c_1, \ldots, c_n)$ have identical type functions $T(f) = T(g)$, then any closed function clone containing $\mathrm{Aut}(\Delta)$ contains f iff it contains g. Since $(\Delta, c_1, \ldots, c_n)$ is homogeneous in a finite language, the type functions are finite objects: $T(f)$ is completely determined by its values on the types of tuples of length q, where q is the maximal arity of a relation in $(\Delta, c_1, \ldots, c_n)$; moreover, there are only finitely many types of q-tuples since $(\Delta, c_1, \ldots, c_n)$ is ω-categorical. As finite objects, these type functions can effectively be used in algorithms [13].

An example of an application of Theorem 7 is the following.

Theorem 8 (Bodirsky, Pinsker and Tsankov [18]). *Let Δ be an ordered homogeneous Ramsey structure in a finite relational language, and let Γ be a reduct in a finite language. Then there exist $c_1, \ldots, c_n$, $m \geq 1$, and m-ary canonical functions $f_1, \ldots, f_k$ on $(\Delta, c_1, \ldots, c_n)$ such that for all reducts Γ' we have that $\mathrm{Pol}(\Gamma') \setminus \mathrm{Pol}(\Gamma)$ is either empty or contains one of the functions $f_1, \ldots, f_k$.*

In words, under the above conditions the finite language reducts of Δ can be distinguished by functions which are canonical after adding finitely many constants to the language of Δ. If we assume that that Δ is finitely bounded, which makes Δ in a way finitely representable, then Theorem 8 can even be implemented in an algorithm. This yields the following effective variant of the theorem.

Theorem 9 (Bodirsky, Pinsker and Tsankov [18]). *Let Δ be an ordered homogeneous Ramsey structure which is finitely bounded, and let Γ, Γ' be finite language reducts of Δ. Then the problem whether or not $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Gamma')$, where the relations of Γ and Γ' are given by quantifier-free formulas over Δ, is decidable.*

This gives hope that tractability of CSPs of reducts of Δ is captured by the canonical functions in their polymorphism clones – cf. Section 3.

The modern proof of Theorem 8 (yet unpublished but available on request) is based on a beautiful characterization of the Ramsey property for homogeneous structures which links Ramsey theory with topological dynamics [32]. A topological group $\mathscr{G}$ is called *extremely amenable* iff whenever it acts continuously on a compact Hausdorff topological space X, then this action has a fixed point, i.e., there exists $x \in X$ such that $g(x) = x$ for all $g \in \mathscr{G}$. Let Δ be an ordered homogeneous relational structure. Then Δ is Ramsey iff $\mathrm{Aut}(\Delta)$, viewed as an abstract topological group, is extremely amenable.

3 Open problems

The research questions presented here are all related to Conjecture 5 in one way or another; in fact, one can put them together so that they constitute a systematic program for proving the conjecture. After stating the questions, in Section 4, I will discuss the questions in the context of the conjecture. I emphasize, however, that each of them has its own mathematical value independently of the truth of the conjecture.

The first set of questions concerns the connection between the algebraic and the topological structure of clones in the light of Theorem 2. As for the link to constraint satisfaction, recall that for finite structures Γ, the complexity of $\mathrm{CSP}(\Gamma)$ only depends on the algebraic structure of $\mathrm{Pol}(\Gamma)$, whereas in the ω-categorical setting, one also has to take the topology on $\mathrm{Pol}(\Gamma)$ into consideration (Theorem 4). A first question, which we shall then refine, is the following.

Question 10. *Are there conditions on oligomorphic algebras under which we can drop the continuity condition in Theorem 2?*

One such condition is to allow only very simple algebras $\mathfrak{B}$. Of particular interest are, of course, continuous clone homomorphisms to $\mathbf{1}$ (i.e., term clones of algebras all of whose functions are projections), since they are the major source of hardness proofs. In fact, we believe that for certain structures they are the *unique* source of hardness proofs: as already mentioned, the finite tractability conjecture states that if a finite structure Γ satisfies a cosmetic condition, then $\mathrm{CSP}(\Gamma)$ is NP-hard if and only if there exists a homomorphism from $\mathrm{Pol}(\Gamma)$ to $\mathbf{1}$. That condition, which requires that $\mathrm{Pol}(\Gamma)$ is *idempotent*, i.e., all $f \in \mathrm{Pol}(\Gamma)$ satisfy the equation $f(x, \dots, x) = x$, can always be assumed: every CSP of a finite structure is equivalent to a CSP of a finite structure with an idempotent polymorphism clone [21]. This fact can be derived in two steps: first, one shows that Γ can be assumed to be a *core*, i.e., all endomorphisms of Γ are automorphisms. One then shows that it is possible to add finitely many constants to the language of Γ without increasing the complexity of its CSP – adding one constant for each element of the domain, this forces all polymorphisms to be idempotent. In the ω-categorical setting, one cannot simply assume that $\mathrm{Pol}(\Gamma)$ be idempotent; indeed, it would then certainly fail to be oligomorphic, containing no unary functions at all except the identity. However, it is possible to perform an analog of the first step, and assume that Γ is a *model-complete core*, meaning that $\mathrm{Aut}(\Gamma)$ is (topologically) dense in the endomorphism monoid of Γ (i.e., every endomorphism *locally* looks like an automorphism) [4]. Moreover, as in the finite case, adding finitely many constants to the language of Γ then does not increase the complexity of its CSP [4]. Until now, similarly to the situation for finite templates, we do not know of a reduct of a finitely bounded homogeneous structure which is a model-complete core with an NP-hard CSP, but which has no continuous homomorphism to $\mathbf{1}$ after adding finitely many constants to the language.

Back to the continuity condition, we therefore ask the following.

Question 11. *If a closed oligomorphic clone has a clone homomorphism to* $\mathbf{1}$ *(i.e., it satisfies no non-trivial equations), does it always have a continuous clone homomorphism to* $\mathbf{1}$*? If not, are there further conditions on the clone (model complete core etc.) which imply a positive answer?*

There exists considerable literature about automorphism groups of ω-categorical structures Γ which are *reconstrucible*, i.e., where the topology on $\mathrm{Aut}(\Gamma)$ is uniquely determined by the algebraic group structure; this is for instance the case when $\mathrm{Aut}(\Gamma)$ has the so-called *small index property*, that is, all subgroups of countable index are open. The small index property has for instance been shown for $\mathrm{Aut}(\mathbb{N}, =)$ [24]; for $\mathrm{Aut}(\mathbb{Q}; <)$ and for the automorphism group of the atomless Boolean algebra [42]; the automorphism group of the random graph [29]; for all ω-categorical ω-stable structures [29]; for the automorphism groups of the Henson graphs [27]. The notion of reconstruction makes perfect sense for function clones, and is of importance for our purposes. Call a closed function clone *reconstructible* iff all isomorphisms with other closed function clones are homeomorphisms. Recent research has shown that for some homogeneous ω-categorical structures with a reconstructible automorphism group, the reconstructability carries over to the polymorphism clone of the structure [16].

Question 12. *Let* $\mathscr{C}$ *be an oligomorphic polymorphism clone whose group of invertible unary functions is reconstructible. When can we conclude that* $\mathscr{C}$ *is reconstructible as well?*

We remark that there exists an example of two ω-categorical structures whose automorphism groups are isomorphic as groups but not as topological groups [25], and that this example has recently been expanded to polymorphism clones by David Evans in a yet unpublished note.

It is well-known that every Baire measurable homomorphism between Polish groups is continuous (see e.g. [31]). So let us remark that there exists a model of ZF+DC where every set is Baire measurable [41]. For the structures Γ that we need to model computational problems as $\mathrm{CSP}(\Gamma)$ it therefore seems fair to assume that the abstract algebraic structure of $\mathrm{Aut}(\Gamma)$ always determines its topological structure; consistency of this statement with ZF has already been observed in [34]. Hence, one could hope to find a model of ZF in which polymorphism clones of ω-categorical structures are reconstructible, or in which all homomorphisms of such clones to $\mathbf{1}$ are continuous.

Question 13. *Do oligomorphic polymorphism clones have reconstruction in an appropriate model of ZF? Are all homomorphisms from oligomorphic polymorphism clones to* $\mathbf{1}$ *continuous in an appropriate model of ZF?*

The concept of a *topological clone* appeared as a necessity for formulating Theorem 2; it is indeed natural considering the importance of abstract clones (known in disguise as *varieties*) for universal algebra and the natural presence of topological groups in various fields of mathematics, in particular topological dynamics. It is known that the closed permutation groups on a countable set are precisely those topological groups that are Polish and have a left-invariant ultrametric [2].

Question 14. *Which topological clones appear as closed function clones on a countably infinite set?*

We now turn to the study of polymorphism clones of reducts of homogeneous Ramsey structures. Here, the approach via canonical functions, based on Theorem 7 and the idea that we keep sufficient information about a clone when we add a sufficiently large finite number of constants and then only consider its canonical functions, has proven extremely fruitful. For example, this was the strategy in the Graph-SAT dichotomy classification [13], and in many other applications [40, 15, 18, 12, 14]; confer also Theorems 8 and 9. Generalizing the Graph-SAT strategy, we arrive at the following ideas.

In the following, let Γ be a reduct of a finitely bounded ordered homogeneous Ramsey structure Δ, and let $c_1, \ldots, c_n \in \Delta$. Then the set of finitary canonical functions on $(\Delta, c_1, \ldots, c_n)$ forms a closed function clone, and hence so does the intersection of this clone with $\mathrm{Pol}(\Gamma)$, which we call the *canonical fragment* of $\mathrm{Pol}(\Gamma)$ with respect to $c_1, \ldots, c_n$. By Theorem 8, this canonical fragment still contains considerable information about $\mathrm{Pol}(\Gamma)$. Recall that its functions define functions on the types of $(\Delta, c_1, \ldots, c_n)$; in fact, these "type functions" form a clone on a finite set. We call this clone the *type clone* of $\mathrm{Pol}(\Gamma)$ with respect to $c_1, \ldots, c_n$, and denote it by $T_{c_1,\ldots,c_n}(\mathrm{Pol}(\Gamma))$. There are infinitely many choices for $c_1, \ldots, c_n$, but up to type equivalence, only finitely many for each n since Γ is ω-categorical [28]. In the Graph-SAT dichotomy, these type clones happened to capture the computational complexity of $\mathrm{CSP}(\Gamma)$ [13].

Question 15. *Does the complexity of* $\mathrm{CSP}(\Gamma)$ *only depend on the algebraic structure of its type clones?*

More precisely, in the Graph-SAT classification it turned out that when $\mathrm{CSP}(\Gamma)$ is tractable, then this fact was captured by some canonical polymorphism, which provided the algorithm; in particular, the answer to the following question was positive. It is nourished by the belief that if $\mathrm{Pol}(\Gamma)$ contains a function which implies tractability, and therefore is of use in some algorithm, then this function can be "canonized" and therefore appears in some type clone (cf. Theorem 7).

Question 16. *If* Γ *is tractable, are there necessarily* $c_1, \ldots, c_n \in \Delta$ *such that* $T_{c_1,\ldots,c_n}(\mathrm{Pol}(\Gamma))$ *corresponds to a tractable CSP?*

The following question asks about the converse. The intuition behind it, again true in the Graph-SAT case, is that algorithms for the type clone can be "lifted" back to the original clone.

Question 17. *If there exist* $c_1, \ldots, c_n \in \Delta$ *such that* $T_{c_1,\ldots,c_n}(\mathrm{Pol}(\Gamma))$ *is tractable, is* $\mathrm{Pol}(\Gamma)$ *tractable?*

The following can be seen as a complexity-free variant of the preceding two questions. It is known that if $T_{c_1,\ldots,c_n}(\mathrm{Pol}(\Gamma))$ satisfies non-trivial equations for some $c_1, \ldots, c_n \in \Delta$, then so does $\mathrm{Pol}(\Gamma)$ [17]. Therefore, if some type clone corresponds to a tractable, and if the finite tractability conjecture holds, then $\mathrm{Pol}(\Gamma)$ does not have a homomorphism to $\mathbf{1}$ (and hence, if NP-hard, would have to have another source of hardness). We do not know if the converse is true as well.

Question 18. *If* $\mathrm{Pol}(\Gamma)$ *does not have a homomorphism to* $\mathbf{1}$*, are there* $c_1, \ldots, c_n \in \Delta$ *such that* $T_{c_1,\ldots,c_n}(\mathrm{Pol}(\Gamma))$ *has no homomorphism to* $\mathbf{1}$*?*

We often think of the Ramsey property as an additional property of Fraïssé classes (and indeed, most homogeneous structures are not Ramsey); we do require the property for our methods. However, we do not require the reducts Γ, but only some "base structure" Δ in which they are definable, to be Ramsey. Hence, if the answer to the following question were positive, then our methods would work for all homogeneous structures.

Question 19. *Can every finitely bounded Fraïssé class be extended by finitely many relations to a finitely bounded Fraïssé class which is in addition Ramsey?*

We remark that this question has recently received considerable attention, and in particular the answer is positive for all Fraïssé classes of digraphs – see [33] for further references.

Let us turn to concrete classes of CSPs. Studying those has its own interest, just like the dichotomies of Graph-SAT problems [13] and temporal constraints [9, 8]; moreover they provide sources of examples for the general questions.

The class of finite partial orders is one of the most natural Fraïssé classes, and the answer to the following question would subsume some older results in theoretical computer science, e.g. in [19], and the classification [9]. A basis for a successful complexity classification of Poset-SAT problems has been established very recently in the form of the classification of the closed supergroups of the automorphism group of the random partial order [40].

Question 20. *Classify the complexity of* $\mathrm{CSP}(\Gamma)$*, for all finite language reducts* Γ *of the random partial order. In other words, classify the complexity of Poset-SAT problems.*

The solution to the following would subsume a considerable amount of results in the literature, e.g., completely the papers [30], [6], and some results in [19], [35]. It would moreover require the extension of our methods to *functional* signatures, a venture interesting in itself.

Question 21. *Classify the complexity of* $\mathrm{CSP}(\Gamma)$*, for all finite language reducts* Γ *of the atomless (= random) Boolean algebra.*

4 The infinite tractability conjecture

Each of the above questions has its own interest for the understanding of oligomorphic function clones, oligomorphic algebras, and their connections with constraint satisfaction. However, these questions really are part of a bigger program around Conjecture 5. To make this connection with Conjecture 5 more evident, let me show an example of how a proof of the conjecture could look. Let Γ be a finite language reduct of a finitely bounded homogeneous structure Δ.

- Assume that $\mathrm{Pol}(\Gamma)$ does not have a continuous clone homomorphism to $\mathbf{1}$ (otherwise, $\mathrm{CSP}(\Gamma)$ is NP-hard by Theorem 4).
- If the answer to Question 11 is positive (possibly in an appropriate model of ZF, cf. Question 13), then $\mathrm{Pol}(\Gamma)$ satisfies non-trivial equations.
- Assuming that Question 19 has a positive answer, we can assume that Δ is Ramsey.
- If Question 18 has a positive answer, then some type clone $T_{c_1,\dots,c_n}(\mathrm{Pol}(\Gamma))$ satisfies non-trivial equations.
- Assuming the tractability conjecture for finite templates, $T_{c_1,\dots,c_n}(\mathrm{Pol}(\Gamma))$ corresponds to a tractable CSP.
- Assuming a positive answer to Question 17, $\mathrm{CSP}(\Gamma)$ is then tractable.

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Coloring random and planted graphs: thresholds, structure of solutions and algorithmic hardness

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Random graph coloring is a key problem for understanding average algorithmic complexity. Planted random graph coloring is a typical example of an inference problem where the planted configuration corresponds to an unknown signal and the graph edges to observations about the signal. Remarkably, over the recent decade or two tremendous progress has been made on the problem using (principled, but mostly non-rigorous) methods of statistical physics. We will describe the methods - message passing algorithms and the cavity method. We will discuss their results - structure of the space of solutions, associated algorithmic implications, and corresponding phase transitions. We will conclude by summarizing recent mathematical progress in making these results rigorous and discuss interesting open problems.

Editors' note: The following references were provided to complement the lectures:

L. Zdeborová and F. Krzakala. Phase transitions in the coloring of random graphs. `https://arxiv.org/abs/0704.1269`.

L. Zdeborová and F. Krzakala. Hiding quiet solutions in random constraint satisfaction problems. `https://arxiv.org/abs/0901.2130`

J. Ding, A. Sly, N. Sun. Proof of the satisfiability conjecture for large k. `https://arxiv.org/abs/0901.2130`

We include here reproductions of the handwritten notes Dr Zdeberová used to support her lectures.

3 × 1.5 hours.   Outline   ⓪a

Define:
~~Graph coloring → random & planted ensemble of graphs.~~

45 min ~~Rando~~ → ~~general factor graph~~ setting. (some global constraint problematic)

~~List a couple of related problems & applications~~

optimiz. & CSP
* ~~K-SAT, CSP in general, perceptron, independent set on random graphs.~~
* ~~statistical inference - SBM, compressed sensing, ~~ ...
  ↳ ~~Bayesian inference.~~

~~Basic facts about random graph coloring~~

~~→ bound on SP~~ ~~COL threshold, basic algs. analyzable~~
~~→ spectral alg. for planted one.~~

45 min ~~Derive BP trees to compute marginals.~~
~~general factor graph.~~ ~~(col as a special case)~~

~~⇝ When is this reasonable on non-tree graphs~~

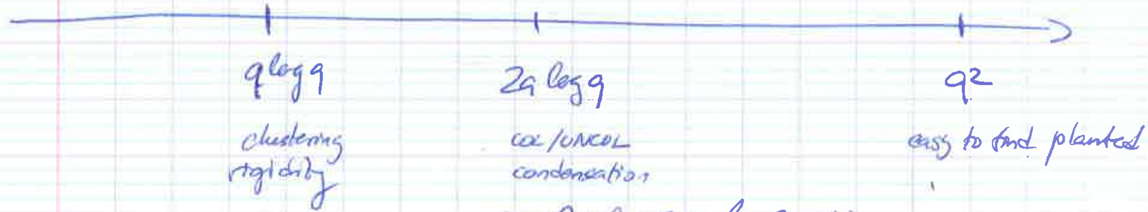
2nd lecture → ~~How to use BP to compute things.~~
~~Quiet planting. How to use planting to understand clusters.~~

→ ~~Structure of solutions, phase transitions & alg. consequences.~~
→ ~~linearized BP + non-backtracking~~

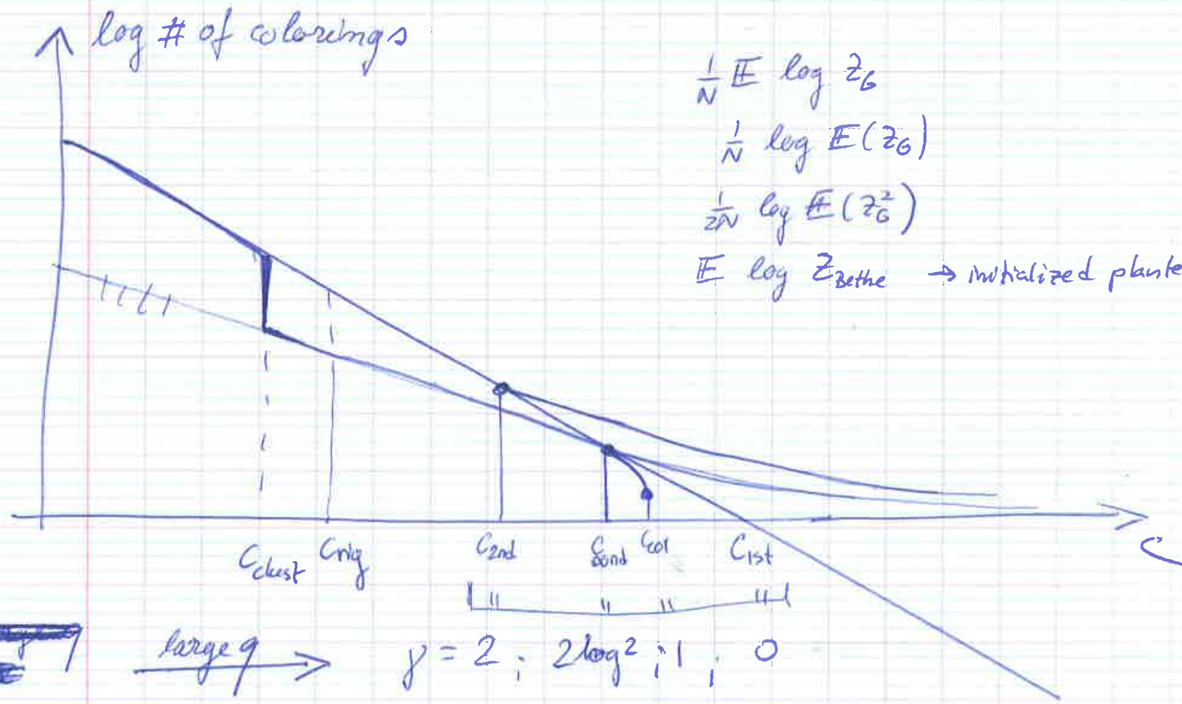
(3rd lecture → ~~results for other problems (?) no time~~
→ ~~Math results~~
→ Finish clusters, $\Sigma(s)$, explain small q the condensed phase
→ Algorithmic consequences. Why do we thing q log q real barrier?
→ Planted ensemble.

Material to keep on the blackboard

$G(N,M)$ q colors, c average degree



$c = 2q\log q - \log q - \gamma$



$\frac{1}{N}\mathbb{E}\log Z_G$

$\frac{1}{N}\log\mathbb{E}(Z_G)$

$\frac{1}{2N}\log\mathbb{E}(Z_G^2)$

$\mathbb{E}\log Z_{Bethe}$ → initialized planted

large q → $\gamma = 2\ ;\ 2\log 2\ ;\ 1\ ;\ 0$

$$\chi^{j\to a}_{s_j} = \frac{1}{Z^{j\to a}} \prod_{b\in\partial j\setminus a} \psi^{b\to j}_{s_j}$$

$$\psi^{a\to i}_{s_i} = \frac{1}{Z^{a\to i}} \sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s_i) \prod_{j\in\partial a\setminus i} \chi^{j\to a}_{s_j}$$

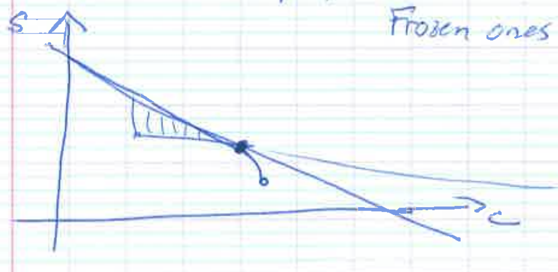
Clusters = equivalence classes w.r.t. BP fixed points initialized in the solutions.

(Geometrical definitions are secondary, not 1 ⟺ 1 ?)

→ For $c < c_{cond}$ majority of solutions behave as planted ~~cluster~~ solution

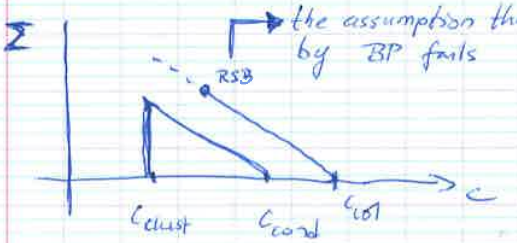
→ Counted # of frozen clusters via SP

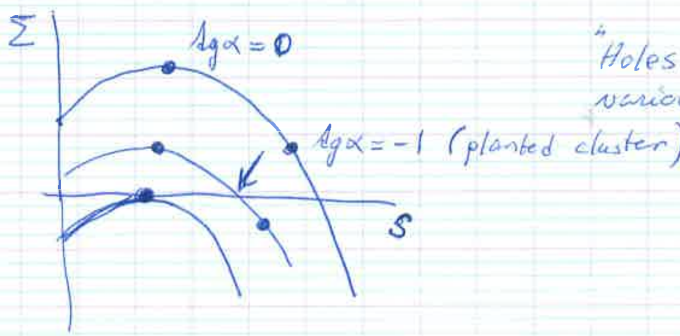
Frozen ones are smaller & more numerous than the equilibrium one.



$\boxed{\text{Zoology of clusters}}$

→ the assumption than the interior gets correctly described by BP fails





"Holes" in the curve Σ or various parts not 1RSB

(→ Solutions found by algorithms are always of that kind.)

Note: ⊛ Many solutions will lead to the trivial BP fixed point.

Those are not described by 1RSB !

⊛ Clusters can cluster. Those the s_{Bethe} will not be a good size.

Still important open problem: Starting from what degree a) all solutions frozen?

b) all solutions in a non-trivial cluster?

Is it qlogq? Ask Amin.

$\boxed{\text{What is this lecture really about?}}$

→ Statistical physics methods for graph theory problems (or computer science more in general problems) and their predictions.

$\boxed{\text{What will we have to overcome together?}}$

→ Language barrier between physics & math. Your help needed
(way of thinking)

$\boxed{\text{What will be the benefit?}}$

→ Many highly non-trivial predictions about well studied problem. Conjectures for you to prove. New working algorithms.

Random graph coloring → our working example, but the methods are much more general. The general framework in ~~few minutes~~. half an hour.

Graph coloring → you know it much better than me.
Given G & q, can the graph be colored with q colors?
If yes, do we have tractable algorithms able to do it?
Why are some instances (graphs) easy and some hard?

Remember will be using all the time

Random graphs: ⊛ (sparse) Erdős-Renyi $G(N,p)$ $p=\frac{c}{N}$ $N\to\infty$, $c=O(1)$
$G(N,M)$ → easier to work with sometimes.
⊛ k-regular ; N nodes, $\frac{Nk}{2}$ edges.
↑degree
⊛ planted random graphs. $Pl(N,M,q)$

→ q groups of N/q nodes. Put M edges at random, but only among different groups.

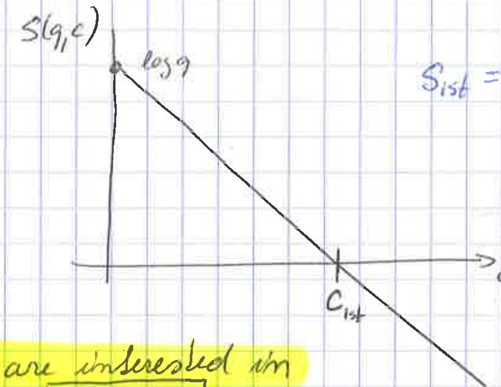
Basic results on random graph coloring:
∘ Friedgut's theorem 1999 & threshold conjecture

Define $Z_G(q)$ the number of proper colorings with q colors of graph G.
$$\mathbb{E}_{G(N,M)}(Z_G(q)) = q^N \left(1-\frac{1}{q}\right)^M$$

Markov's inequality $\underbrace{P(Z_G \geq 1)}_{\text{colorable}} \leq \mathbb{E}(Z_G) = e^{N\log q + M\log(1-\frac{1}{q})}$

$= e^{N[\log q + \frac{c}{2}\log(1-\frac{1}{q})]}$

$S_{1st} = \log q + \frac{c}{2}\log(1-\frac{1}{q})$ $= e^{N S_{1st}(q,c)}$



$$c_{1st} = -\frac{2\log q}{\log(1-\frac{1}{q})} = -\frac{2\log q}{\cancel{1}-\frac{1}{q}-\frac{1}{2q^2}+\left(\frac{1}{q^3}\right)}$$

leading order $\underset{q\to\infty}{=} \frac{2q\log q}{(1+\frac{1}{2q})+(\frac{1}{q^2})} = 2q\log q - \log q + o(1)$

We are interested in $\boxed{N\to\infty}$

$\Rightarrow$ for $c > c_{1st}$ with high probability ($P\to 1$ as $N\to\infty$) no coloring.

[For $c < c_{1st}$ $\to$ There might be rare graphs with very many colorings $\Rightarrow$ $\mathbb{E}(Z_G)$ may be large even if $P(Z_G \geq 1)$ is still small.]

The case here!

Paley - Zygmund inequality $\forall N$: $P(Z>0) \geq \frac{\mathbb{E}(Z)^2}{\mathbb{E}(Z^2)}$
(Consequence of Cauchy - Schwarz)

Hence if we show that $\mathbb{E}(Z^2) < C\,\mathbb{E}(Z)^2$ for some C, with some $C(q,c) \geq 1$ independent of N then

$$P(Z_G \geq 1) \geq \frac{1}{C}$$

Say before

[Friedgut's sharp threshold theorem (for COL with Achlioptas) 1999: $\forall k\geq 3$ $\exists$ $c_{col}(q,N)$ s.t. $\forall \varepsilon > 0$ $G(N,M)$ with $\frac{M}{N} < (1-\varepsilon)c_{col}$ is q-colorable w.h.p. while for $\frac{M}{N} > (1+\varepsilon)c_{col}$ it is not w.h.p

Existence of the limit for coloring $\lim_{N\to\infty} c_{col}(q,N)$ is open. But widely believed to exist.]

Hence $P(Z_c \geq 1) \geq \frac{1}{C}$ & Friedgut $\Rightarrow$ $P(Z_c \geq 1) \underset{N\to\infty}{=} 1$

if $\mathbb{E}(Z^2) < C\,\mathbb{E}(Z)^2$.

Define $S_{2nd} = \frac{1}{2N}\log \mathbb{E}(Z^2)$

→ C_{1st}, C_{2nd}, Armin upper bound
→ true entropy + physics cites
→ recent math results
+ conjecture for SAT & large k proven

With the figure. Cite also who did things in physics.
Mezard, Parisi 1980-90
Method : Mezard, Parisi, Zecchina '02 → sparse case | well done
COL/UNCOL : Mulet, Pagnani, Weigt, Zecchina '02
Cond : Krzakala, Montanari, Ricci-Tersenghi, Semerjian, LZ '07
(Algs. : Zdeborova, Krzakala '07
Planted Col : Krzakala, Zdeborova '09)
Still open for small k ! Far less simple.
→ maybe explain a bit in the last lecture.

Let me go enthousiastic & say that each of these conjectures is worth the fields medal :-).

⊛ Work out this thing about planting 2 and more solutions. Down to what c is it quiet? Down to what c we have $\mathbb{E}(Z^2) \sim \mathbb{E}(Z)^2$?

~~Achlioptas & Moore Naor, prove~~ $C_{2nd} \geq$ ~~$2(q-1)\log(q-1) = 2q\log q - 2\log q - 2 + o(1)$~~

~~Coja-Oghlan '13~~ ~~gap of $(\log q + 1)$~~

~~Moore (conjecture)~~ C_{2nd}

⊛ Bapst, Coja-Oghlan, Rassmann, Vilenchik $k > k_0$ $\boxed{C_{cond}}$

Define C_{2nd} as largest c s.t.

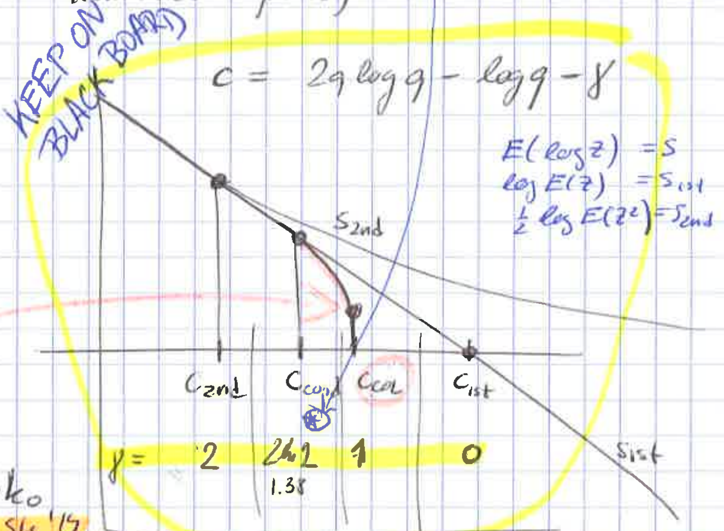
$$\lim_{N\to\infty} \frac{1}{N}\left[\log \mathbb{E}(Z^2) - 2\log \mathbb{E}(Z)\right] = 0$$

Achlioptas & Naor '05 prove $C_{2nd} \geq 2(q-1)\log(q-1) \geq 2q\log q - 2\log q - 2 + o(1)$

C. Moore conjectures '11 $C_{2nd} = 2q\log q - \log q - 2 + o(1)$
(exact in "permuted" coloring ... technical point)

← More citations to physics.

~~Achlioptas~~
Coja-Oghlan '13
upper bound
$\gamma_{col} \geq 1$



Open problem

C_{col} $k > k_0$

→ Solved for SAT Jian, Sun, Sly '14

↦ What about finding the colorings? ↓ ALGORITHMS ↓

Here it becomes interesting.

→ basically assign randomly from what is not on neighbors. & variants.

E.g. in large q, all work only for $c < q\log q$.

Open problem: Poly alg. that finds colorings w.h.p. for $\varepsilon > 0$ $c = (1+\varepsilon)\, q\log q$ or prove impossible

↦ Planted coloring $G^{(N,M)}_{planted} \sim G(N,M)$ for $c < C_{cond}$

Achlioptas, Coja-Oghlan '08, Coja-Oghlan '14

Finding the planted coloring easy for $c >$ const. q^2,

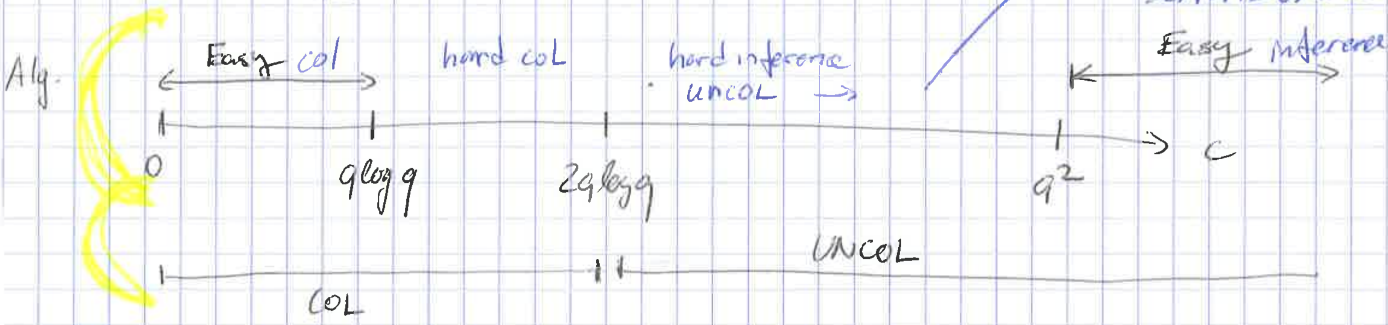
Mossel, Coja-Oghlan, Vilenchik ... 2007

How? Look at spectrum of $\underline{\underline{A}}$.

Open problem: Poly alg that finds the planted cof for $c = (1-\varepsilon)\, q^2$ or prove otherwise impossible

Summary

KEEP ON BLACKBOARD



[Absolutely fascinating problem with open questions that go to the ~~h~~ heart of the deepest problem of computer science
(what is algorithmically ~~possible~~ tractable?)]

Cite from ~~Dian~~ Ding, Sly, Sun paper:

abstract "The satisfiability threshold $\alpha_s(k)$ is given explicitly by the one-step RSB prediction from statistical physics"

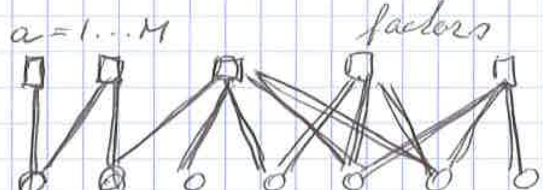
2nd paragraph "Our proof relies heavily on insights from statistical physics, in particular the way in which solutions break into clusters"

Ideally 1/2 of 1st lecture

Introduction to the cavity method. "gentle one"

~~But before~~ Let me state a much more general framework to which the method can be applied. (And some examples of problems.)

Graphical model:



factor graph

nodes, variables

$i = 1 \ldots N$

$$P(\{s_i\}_{i=1\ldots N}) = \frac{1}{Z} \prod_{a=1}^{M} f_a(\{s_i\}_{i\in\partial a})$$

$s_i \in \mathcal{D}$

Examples:

CSP

- Coloring

$\mathcal{D} = \{1, \dots, q\}$ $\quad$ $a = (ij)$ edges

$f_{ij}(s_i, s_j) = (1 - \delta_{s_i, s_j})$ (Z then counts solutions.)

- K-SAT

$s_i \in \{TRUE, FALSE\}$

$fa(\{S_i\}_{i \in \partial a}) = 0$ for one given sequence out of the 2^{k_a}

$|\partial a| = k_a$ $\quad$ 1 otherwise

$(s_1 \vee s_2 \vee \neg s_3) \wedge (s_1 \vee \neg s_5) \wedge (s_3 \vee \neg s_5) \dots$ CNF

+perceptron.

OPTIMIZATION

- independent set largest

$s_i \in \{0, 1\}$ ← does not belong, ← belongs to IS

$f_{ij}(s_i, s_j) = 0$ if $s_i = s_j = 1$
$\quad\quad\quad\quad 1$ otherwise

$f_i(s_i) = e^{\beta s}$ $\quad$ $\beta \to \infty$ the largest IS dominates.

$P(\{s_i\}) = \prod_{i=1}^{N} f_i(s_i) \prod_{(ij) \in E} f_{ij}(s_i, s_j)$

- ~~[illegible]~~ max-cut

$s_i = \{\pm 1\}$

$f_{ij} = e^{-\beta \delta_{s_i s_j}}$ $\quad$ $\beta \to \infty$

statistical inference

- Stochastic block model

q groups, $\frac{c_{ab}}{N}$ prob of connection for nodes from a & b

n_a sizes

Find back the groups from the knowledge of (not just any)

q, c_{ab}, n_a, A

$$P(\{s_i\} \mid A, q, n_a, c_{ab}) = \frac{P(A \mid \{s_i\}) \, P(\{s_i\})}{Z} =$$

$$= \prod_{i=1}^{N} n_{s_i} \prod_{(ij)} \left(1 - \frac{c_{s_i s_j}}{N}\right)^{1-A_{ij}} \left(\frac{c_{s_i s_j}}{N}\right)^{A_{ij}}$$

- Compressed sensing ; measuring directly compressed (signal processing)

$$y_\mu = \sum_{i=1}^{N} F_{\mu i} x_i + w_\mu$$

$$P(x_i \mid F, y) = \prod_i P_i(x_i) \prod_\mu e^{-\frac{(y_\mu - \sum F_{\mu i} x_i)^2}{2\Delta}}$$

SKIP IF NEEDED (if less than 40 min left)

~~Useful to give an example of what cannot be formulated this way (at least not straightforwardly)~~

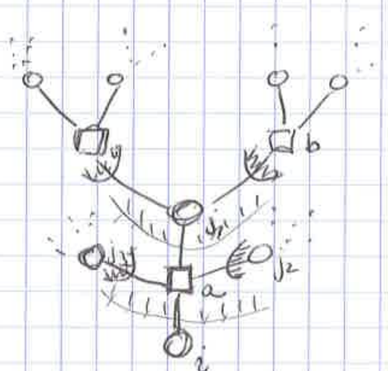
- ~~Hamiltonian path, spanning tree~~
  ~~→ no way around this global constraint with more complicated variables~~
- ~~Planarity of a graph~~
-

Counting solutions, or more in general computing the Z!

In general sharp P-complete → hopeless.

Tractable for special cases. Factor graph with no loops:

Z can be computed iteratively:



Define

$$R_{s_j}^{j\to a} \equiv \sum_{\{s_k\}_{k \text{ above } j}} \prod_{b \text{ above } j} f_b(\{s_\ell\}_{\ell\in\partial b})$$

$$\nu_{s_i}^{a\to i} \equiv \sum_{\{s_j\}_{j \text{ above } i}} \prod_{\substack{b \text{ above } a \\ \& a}} f_b(\{s_\ell\}_{\ell\in\partial b})$$

$$Z = \sum_{\{s_i\}_{i=1..N}} \prod_a f_a(\{s_k\}_{k\in\partial a}) \quad \text{the goal}$$

$$\nu_{s_i}^{a\to i} = \sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s_i) \prod_{j\in\partial a\setminus i} \underbrace{\sum_{\{s_k\}_{k \text{ above } j}} \prod_{b \text{ above } j} f_b(\{s_k\}_{k\in\partial b})}_{R_{s_j}^{j\to a}}$$

$$R_{s_j}^{j\to a} = \prod_{b\in\partial j\setminus a} \nu_{s_j}^{b\to j}$$

Make these "auxiliary normalizations" probabilities

$$\psi_{s_i}^{a\to i} \equiv \frac{\nu_{s_i}^{a\to i}}{\sum_s \nu_s^{a\to i}} \qquad \lambda_{s_j}^{j\to a} \equiv \frac{R_{s_j}^{j\to a}}{\sum_s R_s^{j\to a}}$$

$$\chi_{s_j}^{j\to a} = \frac{\prod_{b\in\partial j\setminus a} \nu_{s_j}^{b\to j}}{\sum_{s}\prod_{b\in\partial j\setminus a}\nu_{s}^{b\to j}} \cdot \frac{\prod_{b\in\partial j\setminus a}\sum_{s''}\nu_{s''}^{b\to j}}{\prod_{b\in\partial j\setminus a}\sum_{s'}\nu_{s'}^{b\to j}} =$$

$$\chi_{s_j}^{j\to a} = \frac{\prod_{b\in\partial j\setminus a}\psi_{s_j}^{b\to j}}{\sum_{s}\prod_{b\in\partial j\setminus a}\psi_{s}^{b\to j}}$$

$$\psi_{s_i}^{a\to i} = \frac{\sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s_i)\prod_{j\in\partial a\setminus i} R_{s_j}^{j\to a}}{\sum_{s}\sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s)\prod_{j\in\partial a\setminus i} R_{s_j}^{j\to a}} \cdot \frac{\prod_{j\in\partial a\setminus i}\sum_{s''}R_{s''}^{j\to a}}{\prod_{j\in\partial a\setminus i}\sum_{s'}R_{s'}^{j\to a}} =$$

$$\psi_{s_i}^{a\to i} = \frac{\sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s_i)\prod_{j\in\partial a\setminus i}\chi_{s_j}^{j\to a}}{\sum_{s}\sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s_j)\prod_{j\in\partial a\setminus i}\chi_{s_j}^{j\to a}}$$

Keep on the black board

$$\chi_{s_j}^{j\to a} = \frac{1}{Z^{j\to a}}\prod_{b\in\partial j\setminus a}\psi_{s_j}^{b\to j} \quad (*)$$

$$\psi_{s_i}^{a\to i} = \frac{1}{Z^{a\to i}}\sum_{\{s_j\}_{j\in\partial a\setminus i}} f_a(\{s_j\}_{j\in\partial a\setminus i}, s_i)\prod_{j\in\partial a\setminus i}\chi_{s_j}^{j\to a} \quad (**)$$

$$Z = \frac{\prod_i Z^i \prod_a Z^a}{\prod_{ia} Z^{ia}} \quad (\Delta)$$

exercise to prove that on trees this is exact.

$$Z^i \equiv \sum_{s_i}\prod_{a\in\partial i}\psi_{s_i}^{a\to i}$$

$$Z^a \equiv \sum_{\{s_i\}_{i\in\partial a}} f_a(\{s_i\})\prod_{i\in\partial a}\chi_{s_i}^{i\to a}$$

$$Z^{ia} \equiv \sum_{s}\psi_s^{a\to i}\chi_s^{i\to a}$$

Marginals (useful quantity)

$$\chi_{s_i}^{i} \equiv \frac{1}{Z^i}\prod_{a\in\partial i}\psi_{s_i}^{a\to i} \quad (\Pi) \quad = \frac{\sum_{\{s_j\}_{j\neq i}}\prod_a f_a(\{s_k\}_{k\in\partial a})}{Z} = \mu^i(s_i)$$

(Really end of 1st lecture)

OK all this might seem heavy, but as long as ~~this is the single graph calculation~~ this is on a tree there is no problem and everything is fully rigorous.

Loopy Belief Propagation algorithm. Pearl '82 Gallager '62

In physics: Bethe, Mezard, Parisi, TAP ...

Estimates Z, & marginals of an arbitrary graphical model by the following iterative scheme

- init $\psi_{s_i}^{a\to i}$; $\chi_{s_i}^{i\to a}$ randomly (normalized)
- iterate the update equations (⊛) & (⊛) (in parallel or sequentially)
- Stop after a certain # of iterations (fixed or conv. crit.)
- Compute Z & $\chi_{s_i}^{i}$ from (Δ) & (□)

Issues? → dependence on initial conditions
dependence on update order
& non-convergence

Even if none of above one has in general $\chi_{s_i}^{i} \neq \mu^i(s_i)$

In fact it turns out to be an extremely useful tool. End of 1st lecture

Example: graph coloring $s_i \in \{1, 2, \ldots q\}$

$$Z = \sum_{\{s_i\}_{i=1..N}} \prod_{(ij)\in E} (1-\delta_{s_i s_j})$$

$$\psi_{s_i}^{(ij)\to i} = \frac{1}{Z^{(ij)\to i}} \sum_{s_j} (1-\delta_{s_i s_j}) \chi_{s_j}^{j\to(ij)} = \frac{1}{Z^{(ij)\to i}} \left[1-\chi_{s_i}^{j\to(ij)}\right]$$

$$\chi_{s_j}^{j\to(ij)} = \frac{1}{Z^{j\to(ij)}} \prod_{k\in\partial j\setminus i} \psi_{s_j}^{(kj)\to j}$$

+ simplify notation

$$\boxed{\chi_{s_j}^{j\to i} = \frac{1}{Z^{j\to i}} \prod_{k\in\partial j\setminus i} \left(1-\chi_{s_j}^{k\to j}\right)}$$

$$\chi^i_{s_i} = \frac{1}{Z^i} \prod_{k\in\partial i} \left(1-\chi^{k\to i}_{s_i}\right)$$

$$Z^i = \sum_s \prod_{k\in\partial i} \left(1-\chi^{k\to i}_s\right)$$

$$Z^{ij} \equiv 1 - \sum_s \chi_s^{i\to j} \chi_s^{j\to i}$$

$$Z = \prod_{i\in G} Z^i \left(\prod_{(ij)\in E} Z^{ij} \right)^{-1}$$

[Proof that this is equivalent to the general formulation for coloring is an exercise.

Factorized (uniform) fixed point :

$$\chi^{i\to j}_{s_i} = \frac{1}{q} \qquad \frac{\left(1-\frac{1}{q}\right)^{d_j-1}}{\sum_s \left(1-\frac{1}{q}\right)^{d_j-1}} = \frac{1}{q}$$

indeed fixed point on every graph

$$Z^i = q\left(1-\frac{1}{q}\right)^{d_i}$$

$$Z^{ij} = 1-\frac{1}{q}$$

$$\log Z_G = \sum_i \left[\log q + d_i \log\left(1-\frac{1}{q}\right)\right] - \sum_{(ij)\in E} \left(1-\frac{1}{q}\right) =$$

$$= N\log q + Nc\log\left(1-\frac{1}{q}\right) - \frac{Nc}{2}\left(1-\frac{1}{q}\right) =$$

$$= N\left[\log q + \frac{c}{2}\log\left(1-\frac{1}{q}\right)\right] = N S_{1st} = \log \mathbb{E}(Z_G)$$

Surely not true for every graph! (on trees it must be true! that we proved here)

Lets see if it is useful somehow on random graphs

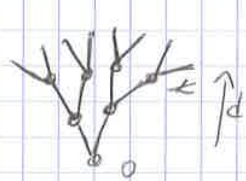
Property : for a fixed c, the shortest loop going trough a node has length $O\left(\frac{\log N}{\log c}\right)$. TREE-LIKE

[For $N\to\infty$ the neighborhood of almost every node is a tree. (up to distance $\frac{\log N}{\log c} \to \infty$).

Back to trees. Let us think about dependence on the boundary condition. (Weak dependence gives hope for reasonability of the assumption)

$\{\psi_{s_i}^{i\to j}\}$ boundary

ψ_s^0



Reconstruction on trees

a) ψ_s^0 does not depend on $\{\psi_{s_i}^{i\to j}\}$ as $d\to\infty$ (the unicity regime)

b) ψ_s^0 does not depend on $\{\psi_{s_i}^{i\to j}\}$ for most boundaries. (the extremality regime)

Most in what measure? One that would correspond to a typical coloring on a tree. Assuming that typical coloring is described by the $\frac{1}{q}$ fixed point (which it is on a tree) this corresponds to generation of the boundaries by the simple process of assigning children differently from parent but randomly otherwise.

c) ψ_s^0 depends on the boundary! even when $d\to\infty$

(Define universality classes of messages in the neighborhood of 0 ~~ato~~ of boundaries)

Within the class → no dependence (extremal measures)

→ clusters

Determine a boundary between regime b) & c):

① Choose a coloring among all colorings at random. s_i^*

② Initialize $\chi_{s_i}^{i\to j} = \delta_{s_i, s_i^*}$

③ Iterate BP till convergence.

fixed point: back at $\frac{1}{q}$ ⇒ extremal regime b)

stays away from $\frac{1}{q}$ ⇒ regime c)

Problem: How to choose a random coloring on a random graph?

Saving the day

Recall planting: Generates a different graph ensemble.
How different.
Planting generates more often graph with more solutions.
In fact $P(G) \sim Z_G$. If Z_G the same on every graph then planting generates the same ensemble as $G(N,M)$.

OK up to c_{cond}, this is enough.

Recall our result that $\log Z_{Bethe} = S_{1st}$ $\forall G$
If $\boxed{\mathbb{E}(\log Z) = \log \mathbb{E}(Z) + o(1)}_{\Delta}$ $\Rightarrow$ planting $\sim G(N,M)$
in the sense that high prob. properties must hold on both.
In consequence the planted conf. = equilibrium conf.

Coja-Oghlan '14 proves (Δ) for $c < c_{cond}$
Bapst, Efthymiou

$\downarrow$ defined by failure of this

$c > c_{cond}$

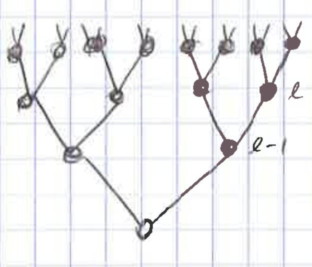
$$\frac{1}{N}\mathbb{E}(\log z) < \frac{1}{N}\log \mathbb{E}(z)$$

To find c_{clust}: Plant, init BP in planted solution & see if fixed point $\frac{1}{q}$ or not when initialized planted

Interesting upper bound: (naive reconstruction)

Prob. that random boundary implies the root:
Probability that the far away boundary is compatible with only one value of the root.

η_ℓ → prob. that a variable in the ℓ^{th} generation is directly implied ~~conditioned on the value of its parent~~ by above nodes. (k-regular graph)



∞ $\eta_\infty = 1$

P(#r of the q-1 colors are not present on in an implied way the k-1 above nodes)

$\left(1 - \frac{r}{q-1}\eta_\ell\right)$ → prob that on one neighbor r of the q-1 colors are not implied

$c_{\text{naive } d}$

Naive reconstruction → boundary between b) & c) lets call that $c_d \le c_{\text{naive } d}$

Consider regular tree. Pick a configuration at random by iterating from the root

Probability that the boundary implies the root

n_ℓ → prob. that a node in the l'th generation is ~~compatible with only one value~~ implied. (can take only its one color)

= all $(q-1)$ colors are ~~represented~~ implied in $(l+1)$

= 1 − at least one of the $(q-1)$ is not forbidden from above.

Fix a color: prob. it is not implied above

$(q-1)$ options.

$$(q-1)\left(1-\frac{n_\ell}{q-1}\right)^{k-1}$$

we need to look at the right color

probability $\frac{1}{q-1}$ to hit it

overcounting! 2 colors are not implied

$$\binom{q-1}{2}\left(1-\frac{2n_\ell}{q-1}\right)^{k-1}$$

$$\boxed{n_\ell = 1 + \sum_{r=1}^{q-1} (-1)^r \binom{q-1}{r}\left(1-\frac{r}{q-1}\, n_{\ell+1}\right)^{k-1}}$$

Analysis to understand this for $n_\infty = 1$

or mathematica

$n_\ell = 0$ always a fixed point. but $n_\ell > 0$

reached for $c > c_{\text{naive}} = \boxed{q \log q} + q \log\log q + 1$.

Ueen!

(12)

$$\sum_{r=0}^{q-1} (-1)^r \binom{q-1}{r} \left(1 - \boxed{\frac{r}{q-1}}\, n_\ell\right)^{k-1} = n_{\ell-1}$$

ways to choose r colors, prob. the corresp. color

↑ ways to choose the r among $q-1$.

directly implied $= (q-1)$ implied above.

$P = 1 - P$(at least one not implied)

$$n_{\ell-1} = 1 - \underbrace{\text{not implied}}_{\text{at least one not forbidden}} \quad (q-1)\left(1 - \frac{1}{q-1} n_\ell\right)^{k-1} + \binom{q-1}{2}\left(1 - \frac{2}{q-1} n_\ell\right)^{k-1} \dots$$

telescopic sum

at least one not forbidden $=$

1 forbidden $-$ 2 f. $+$ 3 f. ...

Rearrangements
expander $\Rightarrow$ to change a frozen spin $O(N)$ have to be flipped !

$n_\infty = 1$ iterate $=$ define C_{rig}.
Not so hard to analyse. Exercise!

Analyse the large q limit of this: $C_{rig} = q\log q + q\log\log q + 1 + o(1)$

← Now it starts to be interesting → Up to here definitely the 2nd lecture.

~~Frozen clusters of solutions!~~

(On a random graph BP is exact in the limit $N \to \infty$
as long as we consider all its fixed points.)

BP fixed point $=$ cluster (except putting away the $\frac{1}{q}$ one when needed)

Fixed point with frozen messages $=$ frozen cluster
(easier to deal with ... large q, k ...)

1RSB $=$ BP on ~~a problem~~ BP fixed points.

variables $\chi^{i\to j}_{s_i}$

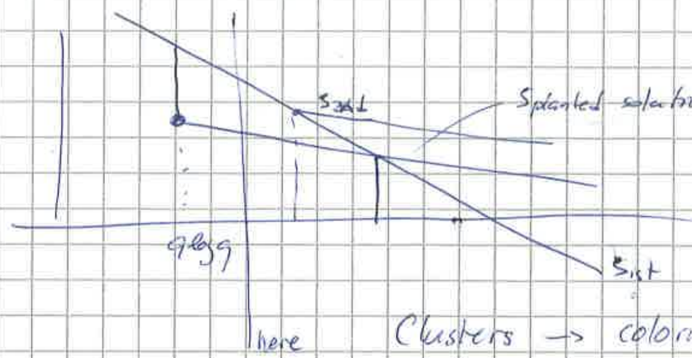
constraints BP equations

weights Z^i, Z^{ij} ...
according to size of clusters

Survey Propagation $=$ BP on frozen BP fixed points weighting all clusters equally.
(Mezard, Parisi, Zecchina '02)

What is going on ?

Planted solution is a typical solution. So how do things look around it? Bethe entropy of the fixed point.



Clusters → colorings spit into many pieces, each looks just like the one around the planted ~~state~~ conf

How to describe these clusters? (Independently of planting)

~~Think~~ Cluster = BP fixed point. (Hypothesis)

Count frozen BP fixed points.

right correlation decay has to hold.

New (CSP) Page R.5

→ Col for regular graphs.

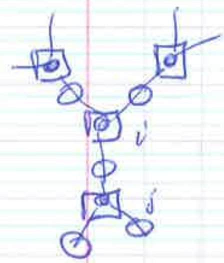
Comment on the hardness of the col regime !

Consider a graph. Variables on edges.

$\underline{q+1 \text{ possible values}}$ of the belief $\left.\begin{matrix}0\\1\\0\\0\end{matrix}\right\}$ on every color

+ the rest.

Constraints on variables on every node



For every direction:

⊛ There cannot be frozen messages on the incoming side for every color.

⊛ If $(q-1)$ colors covered by the incoming frozen messages then the qth one is implied

⊛ If at least two colors are not frozen on the incoming side $\Rightarrow$ joker.

$\eta_s^{i\to j}$ = prob. that the $\begin{matrix}0\\0\\\vdots\\1\\\vdots\\0\end{matrix}$ $\rightarrow$ position s is send on (ij)

$s \in \{1,\dots,q,*\}$

$\eta_*^{i\to j}$ the joker

$\nearrow$ s_{ij} and only s_{ij} is not implied on the k's

$$\eta^{i\to j}_{s_{ij}} = \frac{1}{Z^{i\to j}} \sum_{\{s_{ki}\}_{k\in\partial i\setminus j}} C\left(\{s_{ki}\}_{k\in\partial i\setminus j},\, s_{ij}\right) \prod_{k\in\partial i\setminus j} \eta_{s_{ki}}^{k\to i}$$

$s \neq *$

$$\cong \prod_{k\in\partial i\setminus j}\left(1-\eta_{s_{ij}}^{k\to i}\right) - \sum_{p\neq s_{ij}} \prod_{k\in\partial i\setminus j}\left(1-\eta_{s_{ij}}^{k\to i}-\eta_p^{k\to i}\right) + \dots$$

$$\dots (-1)^q \prod_{k\in\partial i\setminus j}\left(1-\sum_{s=1}^{q}\eta_s^{k\to i}\right)$$

$Z^{i\to j}$ = no contradiction = at least one is not implied on the neighbors.

~~at least two colors are not implied~~

$$= \sum_p \left(1-\eta_p^{k\to i}\right) - \sum_{p,r} \prod\left(1-\eta_p-\eta_r\right) \dots -(-1)^q \prod\left(1-\sum_1^q \eta_s\right)$$

These are the SP equations

End of 2nd lecture

For symmetric colors & regular graphs page 14 + corresponding expression for Z.

Summarizing clustering

Last time (refer to my figure with the entropies)
- naive reconstruction bound:
  Planted conf. is in a frozen cluster $\Rightarrow q \log q$
- survey propagation: counting frozen clusters
  $\Rightarrow$ col / uncol

How to describe ~~more~~ clusters in more detail.

BP fixed point = cluster
Σ its size (external entropy)

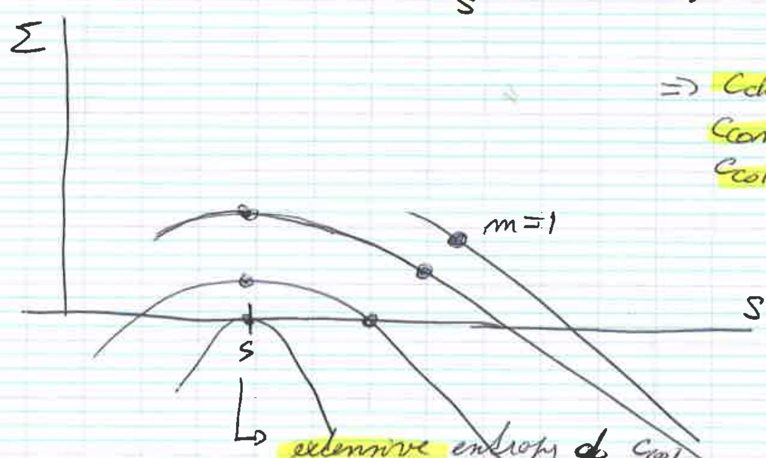
variables = messages, constraint = message passing fixed point

1RSB = BP over BP (generic - non-frozen - fixed points can be done numerically)

Define $\Sigma(s)$

$$\max_s \left(\Sigma(s) + s \right) \Leftrightarrow \Sigma'(s) = -1$$

$\Rightarrow$ c_{clust}; planted cluster
c_{cond} $E \log Z < \log E Z$
$c_{col/uncol}$



$\hookrightarrow$ extensive entropy at c_{col}

Condensed phase $\rightarrow$ exp. many clusters, but to cover almost all solution only $O(1)$ needed

Small q: $q=3$ $c_{clust} = c_{cond} = c_{planted}$
$q>3$ $>$ $>$

Rigidity starts close to c_{col} at $q=3$, moves forward

Algorithms to look for coloring in random col

- Simulated annealing
- BP decimation & reinforcement → highly heuristic
- SLS
- Survey propagation decimation / reinforcement

For small q very good.

Observations: Solutions that are found by algorithms have very

⊛ different properties from the planted (equilibrium) solution

→ they do not have an associated BP fixed point. :-(

Frozen solutions never found !

⊛ Rearrangment are large with frozen variables.

⊛ ~~Annealing~~ In problem with only frozen clusters none of these algorithms work.

⇒ Conjecture: These intuitions gained from BP & mere like algorithms applies more generically.

Nothing works when all clusters frozen.

$c \sim (1+\varepsilon)\, q \log q$

Maybe Nothing works when all solutions have non-trivial fixed points.

→ go on page 14

~~Algorithmic consequences:~~ (conjectures)

- ~~Sampling gets hard at c_{clust} (from almost all)~~
- ~~Finding solutions gets hard when all solutions are in frozen clusters, $\sim c_{rig}$~~

MAYBE ELSEWHERE

Value of the threshold for regular graphs:

~~If more material needed I can derive this.~~ derived page 12.5

$$\eta = \frac{\sum_{l=0}^{q-1} (-1)^l \binom{q-1}{l} \left(1-(l+1)\eta\right)^{d-1}}{\sum_{l=0}^{q-1} (-1)^l \binom{q}{l+1} \left(1-(l+1)\eta\right)^{d-1}}$$

fixed point

$$\Sigma = \log \sum_{l=0}^{q-1} (-1)^l \binom{q}{l+1} \left(1-(l+1)\eta\right)^{d-1} - \frac{c}{2} \log\left(1-q\eta^2\right)$$

$\leadsto$ where this goes to zero.

For small q. ~~$c_{2nd} < c_{cond}$~~ $c_{clust} \leq c_{cond} < c_{col}$

- ~~$q=3$ one has $c_{clust} = c_{cond}$~~
- ~~c_{rig} is close to c_{col} for small q!~~

Lets go back to the planted ensemble. recap: impossible / hard / ease

We still have the problem of $c=(1-\varepsilon)q^2$.

Coja-Oghlan et al '08-14
US '2009 (non-rigorous)

planted = random
$c \leq c_{cond}$.

Notably for $c_{cond} > c > c_{clust}$ the planted cluster is just one of exp. many. Indistinguishable.

For $c > c_{cond}$ nothing changes only the planted cluster is added to the space of solutions;

And algorithmically? Very interesting!

Exhaustive enumeration ok for $c > c_{cond}$. But tractably?

Even in the planted ensemble $\chi_{s_0}^{i\to j} = \frac{1}{q}$ is a fixed point.
Is it linearly stable ?

$$\chi_{s_i}^{i\to j} = \frac{1}{Z^{i\to j}} \prod_{k\in\partial i\setminus j} \left(1-\chi_{s_i}^{k\to i}\right) = \frac{\prod_{k\in\partial i\setminus j}\left(1-\chi_{s_i}^{k\to i}\right)}{\sum_s \prod_{k\in\partial i\setminus j}\left(1-\chi_s^{k\to i}\right)}$$

$$\left.\frac{\partial \chi_{s_i}^{i\to j}}{\partial \chi_{s_k}^{k\to i}}\right|_{\frac{1}{q}} = -\frac{\delta_{s_i s_k} \prod_{l\in\partial i\setminus k,j}\left(1-\chi_{s_i}^{l\to i}\right)}{\sum_s \prod_{k\in\partial i\setminus j}\left(1-\chi_s^{k\to i}\right)} + \frac{\prod_{k\in\partial i\setminus j}\left(1-\chi_{s_i}^{k\to i}\right) \prod_{l\in\partial i\setminus j,k}\left(1-\chi_{s_k}^{l\to i}\right)}{\left[\sum_s \prod_{k\in\partial i\setminus j}\left(1-\chi_s^{k\to i}\right)\right]^2} =$$

$$= -\delta_{s_i s_k} \frac{\left(1-\frac{1}{q}\right)^{d_i-2}}{q\left(1-\frac{1}{q}\right)^{d_i-1}} + \frac{\left(1-\frac{1}{q}\right)^{d_i-1}\left(1-\frac{1}{q}\right)^{d_i-2}}{q^2\left(1-\frac{1}{q}\right)^{2(d_i-1)}} =$$

$$= -\delta_{s_i s_k}\frac{1}{q\left(1-\frac{1}{q}\right)} + \frac{1}{q^2\left(1-\frac{1}{q}\right)} = \frac{1-q\,\delta_{s_i s_k}}{q(q-1)}$$

$$\chi_{s_i}^{i\to j} = \frac{1}{q} + \varepsilon_{s_i}^{i\to j}$$

out diagonal $\frac{1}{q(q-1)}$

diagonal $-\frac{1}{q}$

$$\varepsilon_{s_i}^{i\to j}(t+1) = \sum_{s_k}\sum_{k\in\partial i\setminus j} \varepsilon_{s_k}^{k\to i}(t)\, \underbrace{\frac{1-q\,\delta_{s_i s_k}}{q(q-1)}}_{T_{s_i s_k}}$$

~~$\frac{1}{q}\left(\frac{1-\delta_{s_i s_k}}{q-1}\right) - \ldots = \frac{1-q\delta_{s_i s_k}}{q(q-1)}$~~ OK

Eigenvalues

$\lambda_2 = 0 \quad [v = (1,1,\ldots,1)]$

$\lambda_1 = \frac{1}{1-q} = -\frac{1}{q-1}$

$(q-1)$ degenerate

$[1\ {-1}\ 0\ 0\ 0 \ldots 0]$

$-\frac{1}{q} + \frac{1}{q(q-1)} = \frac{-q}{q(q-1)}$

$\frac{1}{q} + \frac{1}{q(q-1)} = -$

$$\vec{\varepsilon}(t+1) = \underset{2M\times 2M}{B} \otimes \underset{q\times q}{T}\ \vec{\varepsilon}(t)$$

↑ the non-backtracking matrix.

Perturbation of message on the root by the leaves.
Mean zero. Variance

$$\mathrm{var}_{root} \simeq c^d \lambda^{2d}\, \mathrm{var}_{leaf}$$

$c = (q-1)^2$ stability criterium

Planted & Random ensemble

Krzakala, Zdeborova 2009

$c < (q-1)^2$ the fixed point $\frac{1}{q}$ stable

Random ens. $c > (q-1)^2$ BP does not converge.

Plante ens. $c > (q-1)^2$ BP converges to the planted conf.

The non-backtracking

Even more interesting

$$\vec{\varepsilon} = B \otimes T \vec{\varepsilon}$$

our PNAS '13

~~Spectrum~~ Top q eigenvectors of B encode the planted configuration as well.

Spectral clustering → B the right operator for clustering of sparse graphs (but also for solving planted CSP).

Take home message: Stat. phys. not only provides intrigueing conjectures & deep insight, but also inovative algorithmic ideas.