Polynomial graph invariants from graph homomorphisms

Delia Garijo[1] **Andrew Goodall**[2] Jarik Nešetřil[2]

[1]University of Seville, Spain

[2]Charles University, Prague

Midsummer Combinatorial Workshop XIX
29 July - 2 August 2013, Prague

Overview

Graph polynomials with a name for themselves...

- **chromatic polynomial**, $P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$
- **Tutte polynomial** (universal for recurrence in $\backslash uv$ and $/uv$)
- **Averbouch–Godlin–Makowsky polynomial** (recurrence in $\backslash uv$, $/uv$ and $-u-v$), includes **matching polynomial**
- **Tittmann–Averbouch–Makowsky** polynomial (recurrence in $\backslash v$, $/v$ and $-N[v]$), includes **independence polynomial**

... polynomials determined by counting H_k-colourings of a graph for a sequence of (multi)graphs $(H_k : k = 1, 2, \dots)$
e.g. for $k \in \mathbb{N}$, $P(G;k)$ counts K_k-colourings

Definition

Graphs G, H.
$f : V(G) \to V(H)$ is a *homomorphism* from G to H if
$uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$.

Definition

H with adjacency matrix $(a_{s,t})$, $a_{s,t}$ weight on $st \in E(H)$,

$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

$$\begin{aligned}\hom(G, H) &= \#\{\text{homomorphisms from } G \text{ to } H\} \\ &= \#\{H\text{-colourings of } G\}\end{aligned}$$

when H simple ($a_{s,t} \in \{0, 1\}$) or multigraph ($a_{s,t} \in \mathbb{N}$)

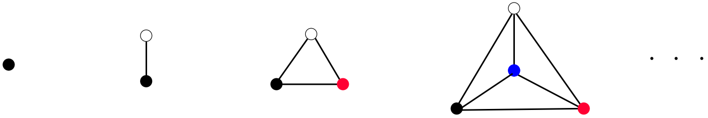
The main question

For sequence $(H_{k,\ell,\dots})$, when is, for all graphs G,

$$\hom(G, H_{k,\ell,\dots}) = p(G; k, \ell, \dots)$$

for polynomial $p(G)$?

Examples

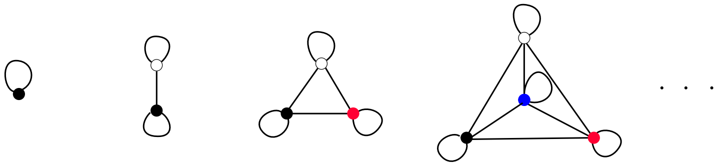


$$(K_k)$$

$$\mathrm{hom}(G, K_k) = P(G; k)$$

chromatic polynomial

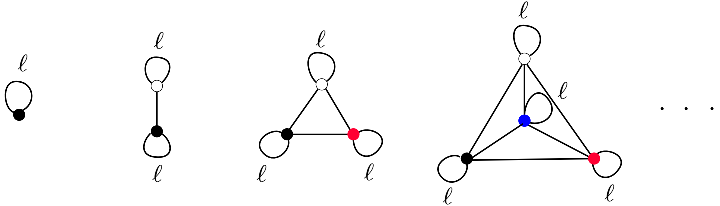
Examples



$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

Examples



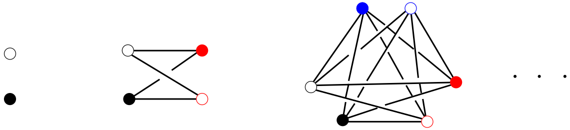
$$(K_k^\ell)$$

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= k^{c(G)}(\ell-1)^{r(G)}T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$

It has something to do with automorphisms...

Examples of strongly polynomial (H_k) so far have $\mathrm{Aut}(H_k) = \mathrm{Sym}_k$.

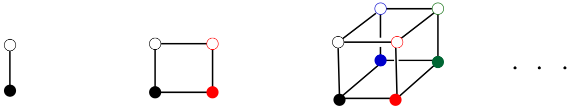


$$(\overline{kK_2}) = (K_{2,\ldots,2})$$

$$\mathrm{Aut}(K_{2,\ldots,2}) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

$$\mathrm{hom}(G, K_{2,\ldots,,2}) = 2^{|V(G)|}P(G;k)$$

... but what precisely?



$$(K_2^{\square k}) = (Q_k) \textit{ (hypercubes)}$$

$$\mathrm{Aut}(Q_k) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

Proposition (Garijo, G., Nešetřil, 2013+)

$\hom(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

(H_k) is *polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for sufficiently large k ($k \geq k_0(G)$)

Since $\hom(G_1 \cup G_2, H) = \hom(G_1, H)\hom(G_2, H)$, suffices to consider *connected* G.

Example

- (K_k), (K_k^1). $(\overline{kK_2})$ strongly polynomial in k
- (K_k^ℓ) strongly polynomial in k, ℓ
- (C_k), (P_k) polynomial in k
- (Q_k) not polynomial in k (but in k and 2^k)

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq H_k \\ |V(S)| \leq |V(G)|}} \mathrm{sur}_{\mathsf{V,E}}(G, S)$$

$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V,E}}(G, S) \ \#\{\text{copies of } S \text{ in } H_k\}$$

Assuming G connected, homomorphic image S also connected

Proposition (De la Harpe & Jaeger, 1995)

- (H_k) *strongly polynomial in* $k \Leftrightarrow$ $\forall \text{connected } S \ \#\{\text{subgraphs} \cong S \text{ in } H_k\}$ *is polynomial in* k

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq_{\text{ind}} H_k \\ |V(S)| \leq |V(G)|}} \text{sur}_{\mathsf{V}}(G, S)$$

$$= \sum_{S/\cong} \text{sur}_{\mathsf{V}}(G, S) \, \#\{\text{induced copies of } S \text{ in } H_k\}$$

when H_k simple.

Proposition (De la Harpe & Jaeger 1995)

- (H_k) *strongly polynomial in* $k \Leftrightarrow$ $\forall \text{connected } S \ \#\{\text{subgraphs} \cong S \text{ in } H_k\}$ *polynomial in* k *for all* $k \in \mathbb{N}$
- *can replace* subgraphs $\cong S$ *by* induced subgraphs $\cong S$ *when* (H_k) *simple graphs*

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq_{\text{ind}} H_k \\ |V(S)| \leq |V(G)|}} \text{sur}_V(G, S)$$

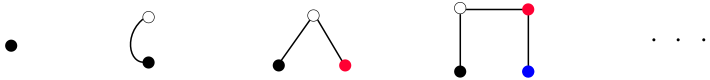
$$= \sum_{S/\cong} \text{sur}_V(G, S) \, \#\{\text{induced copies of } S \text{ in } H_k\}$$

when H_k simple. (for each S want this polynomial in k)

Proposition (De la Harpe & Jaeger 1995)

- (H_k) *strongly polynomial in* $k \Leftrightarrow$ $\forall \text{connected } S \ \#\{\text{subgraphs} \cong S \text{ in } H_k\}$ *polynomial in* k *for all* $k \in \mathbb{N}$
- *can replace subgraphs* $\cong S$ *by induced subgraphs* $\cong S$ *when* (H_k) *simple graphs*

Polynomial but not strongly polynomial



$$(P_k)$$

$$\hom(G, P_k) = \sum_{1 \leq j \leq \min\{|V(G)|, k\}} \mathrm{sur}_V(G, P_j)\,(k-j+1)$$

$\hom(P_4, P_2) = 2$, and $\hom(P_4, P_k) = 8k - 16$ for $k \geq 3$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

If (H_k) strongly polynomial, H_k simple, then

- $(\overline{H_k})$
- $(L(H_k))$

strongly polynomial.
Also, (ℓH_k) strongly polynomial in k, ℓ.

Proposition (Garijo, G., Nešetřil, 2013+)

If (H_k) strongly polynomial, at most one loop each vertex of H_k, then

- (H_k^0) *(remove all loops)*
- (H_k^1) *(add loops to make 1 loop each vertex)*

strongly polynomial.
More generally, (H_k^ℓ) strongly polynomial in k, ℓ.

Proposition

If (F_j), (H_k) *strongly polynomial, then*

- $(F_j \cup H_k)$
- $(F_j + H_k)$

strongly polynomial in j, k.

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$
- complement: (K_k) *(chromatic polynomial)*
- loop-addition: (K_k^ℓ) *(Tutte polynomial)*
- join: $(K_{k-j}^1 + K_j^\ell)$ *(Averbouch–Godlin–Makowsky polynomial)*

$$\hom(G, K_{k-j}^1 + K_j^\ell) = \xi(G; k, \ell-1, -j(\ell-1))$$

Three-term recurrence: for $uv \in E(G)$,

$$\xi(G) = a\xi(G/uv) + b\xi(G\backslash uv) + c\xi(G-u-v)$$

Definition

Given simple graph H, set of graphs $\{F_v : v \in V(H)\}$, the *composition* $H[\{F_v : v \in V(H)\}]$ is formed by

- disjoint union of $\{F_v : v \in V(H)\}$,
- join F_u and F_v whenever $uv \in E(H)$

Proposition (de la Harpe & Jaeger, 1995)

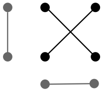
If $(F_{v;k_v})$ strongly polynomial sequence in k_v, each $v \in V(H)$, then $(H[\{F_{v;k_v}\}])$ strongly polynomial in $(k_v : v \in V(H))$.

Example

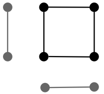
- $K_r[\{\overline{K_{k_1}}, \ldots, \overline{K_{k_r}}\}] \cong K_{k_1,\ldots,k_r}$ (*complete r-partite graph*)
- $F_{v;k_v} = F_k$ all $v \in V(H)$ gives lexicographic product $H[F_k]$

Graph products: direct, cartesian, lexicographic

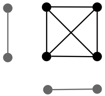
Graphs H, H', $\quad u, v \in V(H)$, $u', v' \in V(H')$



$H \times H'$ $\qquad u \sim u'$ and $v \sim v'$



$H \,\square\, H'$ $\qquad u = v$ and $u' \sim v'$,
or $u \sim v$ and $u' = v'$



$H[H']$ $\qquad u \sim v$,
or $u = v$ and $u' \sim v'$

Proposition (Garijo, G., Nešetřil, 2013+)

If (F_j) *and* (H_k) *strongly polynomial, then*

- $(F_j \times H_k)$
- $(F_j[H_k])$

strongly polynomial in j, k.

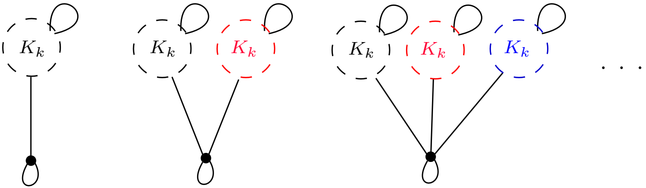
Question

Strongly polynomial:

- $(\overline{K_j} + \overline{K_k}) = (K_{j,k})$
- $(L(K_{j,k})) = (K_j \Box K_k)$ *(Rook's graph)*

If (F_j), (H_k) strongly polynomial, is then $(F_j \Box H_k)$ also?

A new type of strongly polynomial sequence



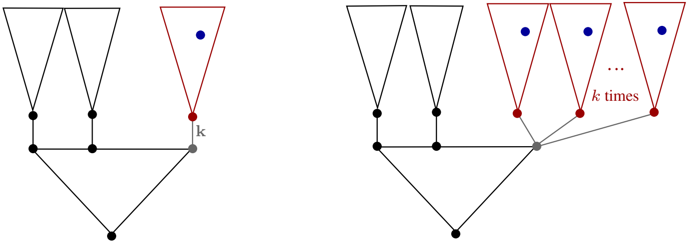
$$H_{j,k} = K_{1,j}[\{K_1^1\} \cup \{K_k^1 \text{ on leaves}\}]$$

$$\hom(G, H_{j,k}) = \sum_{U \subseteq V(G)} j^{c(G[U])} k^{|U|}$$

Tittmann–Averbouch–Godlin polynomial

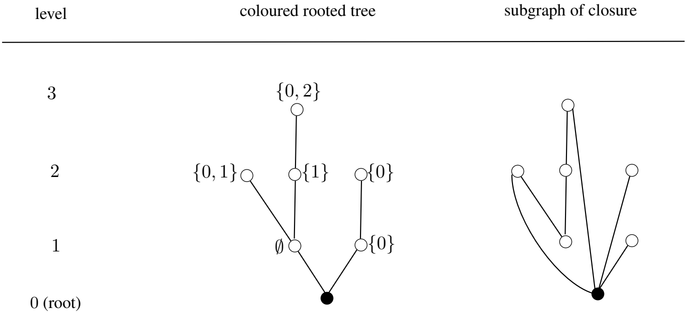
(includes independence polynomial, satisfies three-term recurrence)

Branching coloured rooted trees

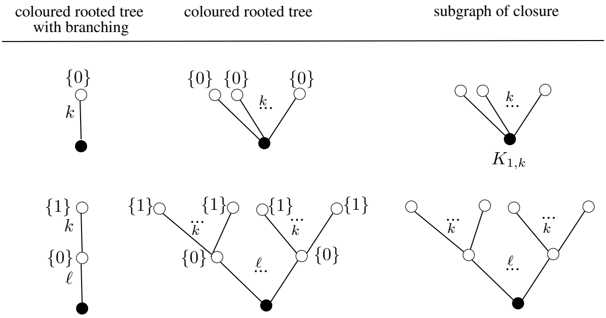


"*k*-branching" at edge of coloured rooted tree

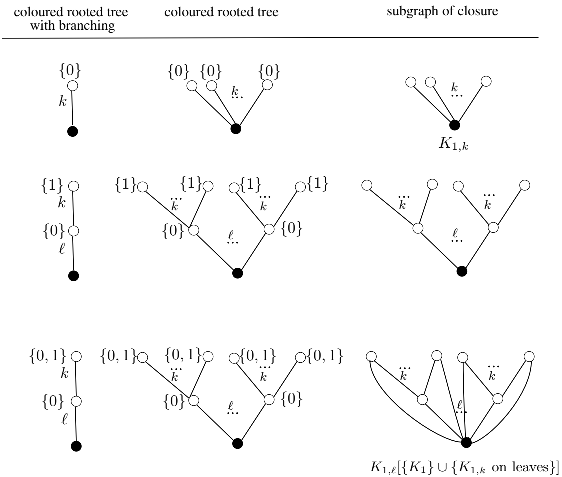
Colours encoding subgraph of closure of rooted tree



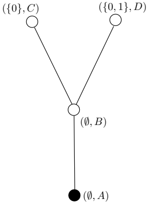
(1) Branching rooted tree encoding subgraph of closure

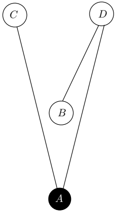


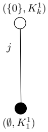
(1) Branching rooted tree encoding subgraph of closure

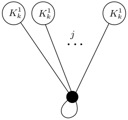


(2) Colours encoding subgraph along with ornaments



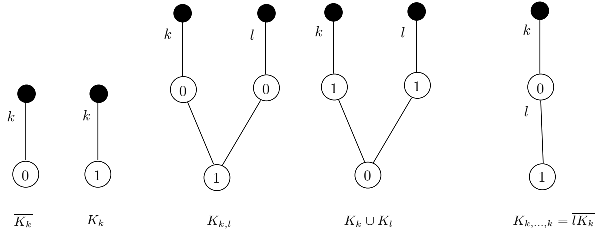






(Tittmann–Averbouch–Makowsky polynomial)

(3) Colours encoding cographs by cotrees



leaves = vertices of cograph

0 = disjoint union, 1 = join

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree T representing graph H*
- *$k, \ell, \ldots$ branching variables on edges of T*
- *after k-branching, ℓ-branching, $\ldots$, obtain coloured rooted tree representing graph $H_{k,\ell,\ldots}$*

Then $(H_{k,\ell,\ldots})$ strongly polynomial in $k, \ell, \ldots$.

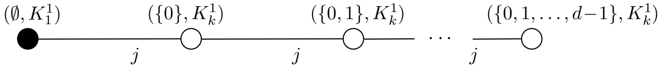
Example

(1) H as a subgraph of closure of T,
colour $s \in V(T) = V(H)$ subset of $\{0, 1, \ldots, \mathrm{height}(T)\}$

(2) ornamented version of (1), strongly poly'l seq. $(F_{s;j_s})$ each vertex $s \in V(H)$, colour as in (1) paired with $F_{s;j_s}$

(3) cotree T encoding of cograph H,
colour non-leaf of T from $\{\cup, +\}$, leaves of $T = V(H)$

coloured rooted tree encoding graph $H_{j,k}$



(ornamented closure of perfect j-ary tree)

$$\hom(G, H_{j,k}) = \sum_{\emptyset \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d \subseteq V} j^{|W_d|} k^{\sum_{1 \leq \ell \leq d} c(G[W_\ell])}$$

Question

This bivariate polynomial generalizes the Tittmann– Averbouch–Makowsky polynomial.
Properties? Evaluations?

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0,1,\ldots,\ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

- Johnson graphs $D = \{k-1\}$
- Kneser graphs $D = \{0\}$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

For every ℓ, D, sequence $(J_{k,\ell,D})$ is strongly polynomial in k.

Question

Can generalized Johnson graphs be generated from simpler sequences by branching coloured rooted trees?

Some further questions

- Is there a characterization of strongly polynomial sequences (H_k) by the sequence of automorphism groups $(\mathrm{Aut}(H_k))$?
- Can (H_k) be verified to be strongly polynomial by testing $\hom(G, H_k)$ for G only in a restricted class of graphs? (yes, for connected graphs – but for a smaller class?)
- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial?