Mathematical Analysis I

Exercise sheet 9

Selected solutions

10 December 2015

References: Abbott 4.5, 4.6, 5.2, Bartle & Sherbert 5.3, 6.1

4. For $q \in \mathbb{Q}$, let $g_q : \mathbb{R} \to \mathbb{R}$ be defined by $g_q(x) = x^q \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $g_q(0) = 0$.

In this question we use the fact that x^q is differentiable on $\mathbb{R} \setminus \{0\}$, with derivative qx^{q-1} when $q \neq 0$ (also at 0 with derivative 0 when $q > 0$), and $\sin\left(\frac{1}{x}\right)$ is differentiable on $\mathbb{R} \setminus \{0\}$ (as the composition of differentiable functions $\sin x$ and $\frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$) and the product and composition rules for derivatives. We also use the result of question 5(ii) that $\sin x$ is differentiable on $\mathbb{R}$ with derivative equal to $\cos x$.

Thus, for $q \neq 0$ and $x \neq 0$,

$$\begin{aligned} g_q'(x) &= x^q \frac{\mathrm{d}}{\mathrm{d}x} \sin\left(\frac{1}{x}\right) + qx^{q-1} \sin\left(\frac{1}{x}\right) \\ &= x^q \left[\left(-\frac{1}{x^2}\right) \cos\left(\frac{1}{x}\right)\right] + qx^{q-1} \sin\left(\frac{1}{x}\right) \\ &= qx^{q-1} \sin\left(\frac{1}{x}\right) - x^{q-2} \cos\left(\frac{1}{x}\right) \end{aligned}$$

When $q = 0$, $g_0'(x) = \left(-\frac{1}{x^2}\right) \cos\left(\frac{1}{x}\right)$ for $x \neq 0$, while by part (i) g_0 is not even continous at 0 so in particular has no derivative at 0.

(i) Show that the function $g_0(x) = \sin\left(\frac{1}{x}\right)$ is not continuous at 0, and that $g_1(x)$ is continuous at 0 but not differentiable at 0.

We show that $g_0(x)$ is not continuous at 0 by producing two sequences (a_n) and (b_n) both convergent to 0 and such that $(g_0(a_n))$ and $(g(b_n))$ converge to different limits. With a view to this, take $a_n = \frac{1}{2\pi n}$ and $b_n = \frac{1}{\pi/2 + 2\pi n}$. Then $g_0(a_n) = 0$ while $g_0(b_n) = 1$.

The function $g_1(x)$ is continuous on $\mathbb{R} \setminus \{0\}$ since it is the product of functions with this property. Continuity at 0 follows from $\lim_{x \to 0} g_1(x) = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 = g_1(0)$. (Use $|x \sin \frac{1}{x}| \leq |x| \to 0$. This is question 3(i).)

(ii) Prove that g_2 is differentiable on $\mathbb{R}$ and calculate $g_2'(x)$. Show that $g_2'(x)$ is discontinuous at 0.

The derivative of g_2 at 0 exists as it is given by

$$\begin{aligned} g_2'(0) &= \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} \\ &= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) \\ &= \lim_{h \to 0} g_1(h) \\ &= 0 \end{aligned}$$

the last equality by part (i), in which it is shown that $g_1(x)$ is continuous at 0.

Hence

$$g_2'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

The limit of $g_2'(x)$ as $x \to 0$ does not exist since while $2x \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0$ the second term $\cos\left(\frac{1}{x}\right)$ does not have a limit as $x \to 0$.

(Proof just as for $\sin\left(\frac{1}{x}\right)$: exhibit two sequences $(x_n) \to 0$ for which $\left(\cos\left(\frac{1}{x_n}\right)\right)$ converges to different limits.)

Hence g_2' is not continuous at 0.

(iii) Find a particular (potentially noninteger) value for q so that

(a) g_q is differentiable on $\mathbb{R}$ but such that g_q' is unbounded on $[0,1]$. We have

$$g_q'(x) = \begin{cases} qx^{q-1}\sin\left(\frac{1}{x}\right) - x^{q-2}\cos\left(\frac{1}{x}\right) & x \neq 0 \\ \lim_{h\to 0} g_{q-1}(h) & x = 0 \quad \text{when the limit exists.} \end{cases}$$

For differentiability of $g_q(x)$ at 0 we require continuity of $g_{q-1}(x)$ at 0, i.e., $q > 1$.
For unboundedness of $g_q'(x)$ on $[0,1]$ we require $q < 2$ (when $x^{q-2}\cos\left(\frac{1}{x}\right)$ is unbounded, while $qx^{q-1}\sin\left(\frac{1}{x}\right)$ is bounded for $q \geq 1$; otherwise both terms are bounded for $x \in (0,1]$).
Hence for $g_q(x)$ to be differentiable on $\mathbb{R}$ while $g_q'(x)$ is unbounded on $[0,1]$ any value $q \in (1,2)$ will serve, e.g. $q = \frac{1}{2}$.

(b) g_q is differentiable on $\mathbb{R}$ with g_q' continuous but not differentiable at 0.
Continuity of $g_q'(x)$ at 0 (and hence all of $\mathbb{R}$) requires $q > 2$ (for $x^{q-2}\cos\left(\frac{1}{x}\right)$ to have a limit as $x \to 0$). For g_q' not to be differentiable at 0 either g_{q-1} is not differentiable at 0 ($q < 2$) or $x^{q-2}\cos\left(\frac{1}{x}\right)$ is not differentiable at 0, which is the case iff $q \leq 3$. (The proof that $x\cos\left(\frac{1}{x}\right)$ is not differentiable at 0 is analogous to showing $g_1(x)$ is not differentiable at 0.)
Hence for any $q \in (2,3]$ the function $g_q(x)$ is differentiable on $\mathbb{R}$ and $g_q'(x)$ is continuous but not differentiable at 0. For example, $q = 3$.

(c) g_q is differentiable on $\mathbb{R}$ and g_q' is differentiable on $\mathbb{R}$, but such that g_q'' is discontinuous at 0.
By the sum, product and composition rules for differentiation, for $x \neq 0$,

$$\begin{aligned} g_q''(x) &= q(q-1)x^{q-2}\sin\left(\frac{1}{x}\right) + qx^{q-1}(-x^{-2})\cos\left(\frac{1}{x}\right) - (q-2)x^{q-3}\cos\left(\frac{1}{x}\right) + x^{q-2}x^{-2}\sin\left(\frac{1}{x}\right) \\ &= q(q-1)x^{q-2}\sin\left(\frac{1}{x}\right) - 2(q-1)x^{q-3}\cos\left(\frac{1}{x}\right) + x^{q-4}\sin\left(\frac{1}{x}\right) \end{aligned}$$

while for $x = 0$,

$$g_q''(0) = \lim_{h\to 0} \frac{g_q'(h) - g_q'(0)}{h} = \lim_{h\to 0}\left[qh^{q-2}\sin\left(\frac{1}{h}\right) - h^{q-3}\cos\left(\frac{1}{h}\right)\right]$$

which exists iff $q > 3$. (The limit is 0 when it exists.)
For $g_q''(x)$ to be discontinuous at 0 we require $q \leq 4$ (so that $x^{q-4}\sin\left(\frac{1}{x}\right)$ is discontinuous at 0; the other terms in the sum above giving $g_q''(x)$ are continuous at 0 for $q > 3$).
Hence we may take any value of $q \in (3,4]$, such as $q = 4$, and g_q will have the required properties.

5.

(i) A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period T if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Prove that if f is a periodic function that is differentiable everywhere then $f'(x)$ is periodic as well. What is the period of f'?

T is a period of f' as well since

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+T+h) - f(x+T)}{h} = f'(x+T+h).$$

Therefore f' is periodic with period T.

(ii) Calculate the derivative of the functions $\sin x$ and $\cos x$. Deduce the derivative of $\tan x$ from these. [*For calculating the derivative of* $\sin x$ *use the identity* $\sin(x+h) = \sin x \cos h + \sin h \cos x$, *while for* $\cos x$ *you might use the analogous identity, or use* $\cos x = \sin(x + \frac{\pi}{2})$.]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \sin x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
&= \lim_{h \to 0} \left[\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right] \\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x,
\end{aligned}$$

where we have used $\lim_{h \to 0} \frac{\sin h}{h} = 1$ from question 3(ii) and

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{-\sin h}{\cos h + 1} \\
&= 1 \cdot 0 = 0
\end{aligned}$$

Using $\cos x = \sin(x + \frac{\pi}{2})$ and the chain rule,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \cos x &= \frac{\mathrm{d}}{\mathrm{d}x} \sin(x + \frac{\pi}{2}) \\
&= \cos(x + \frac{\pi}{2}) \\
&= -\sin x
\end{aligned}$$

Using the rule for differentiating the quotient of two functions,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \tan x &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\sin x}{\cos x} \right) \\
&= \frac{\cos x \cdot \cos x - (-\sin x) \sin x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x
\end{aligned}$$

(iii) Write down domains for the functions sin and cos restricted to which these functions become bijections. Calculate the derivatives of the inverse functions $\sin^{-1}$ and $\cos^{-1}$.

We have bijections $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, $\cos : [0, \pi] \to [-1, 1]$ and $\tan(-\frac{\pi}{2}, \frac{\pi}{2}) \to (-\infty, \infty)$.

Let $x = \sin^{-1} y$, for $y \in [-1, 1]$. Differentiating the identity $\sin x = \sin(\sin^{-1} y) = y$, for

$y \in (-1,1)$,[1] we obtain by the chain rule

$$\begin{aligned} 1 &= \frac{\mathrm{d}}{\mathrm{d}y} \sin(\sin^{-1} y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}(\sin^{-1} y) \cos(x) \end{aligned}$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}y}(\sin^{-1} y) = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

Similarly, differentiating $\cos x = \cos(\cos^{-1} y) = y$ $(y \in (-1,1))$ yields

$$\frac{\mathrm{d}}{\mathrm{d}y}(\cos^{-1} y) = -\frac{1}{\sqrt{1-y^2}}.$$

For $x = \tan^{-1} y$, $y \in \mathbb{R}$, differentiating $y = \tan(\tan^{-1} y)$ yields

$$\begin{aligned} 1 &= \frac{\mathrm{d}}{\mathrm{d}y} \tan(\tan^{-1} y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}(\tan^{-1} y) \cdot \left(1 + \tan^2 x\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}(\tan^{-1} y) \cdot (1 + y^2), \end{aligned}$$

whence $\frac{\mathrm{d}}{\mathrm{d}y}(\tan^{-1} y) = \frac{1}{1+y^2}$.

6. The *hyperbolic functions* sinh, cosh, tanh : $\mathbb{R} \to \mathbb{R}$ are defined for $x \in \mathbb{R}$ by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}.$$

(i) Determine the range of each function sinh, cosh and tanh.

We have $0 < e^x < 1$ for $x < 0$ and $1 \le e^x < \infty$ for $x \ge 0$ (and the exponential function is a bijection $(-\infty, \infty) \to (0, \infty)$. Hence $1 < \cosh x = \frac{e^x+e^{-x}}{2} < \infty$ and $-\infty < \sinh x = \frac{e^x - e^{-x}}{2} < \infty$. Also

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

takes values between $\frac{0+1}{0-1} = -1$ and 1 (the limiting value of $\frac{y-1}{y+1}$ as $y \to \infty$).

Hence $\cosh : (-\infty, \infty) \to (1, \infty)$, $\sinh : (-\infty, \infty) \to (-\infty, \infty)$ and $\tanh : (-\infty, \infty) \to (-1, 1)$.

(ii) Calculate the derivatives of sinh, cosh and tanh. Using $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} e^{-x} = -e^x$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \frac{e^x - e^{-x}}{2} = \sinh x,$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \frac{e^x + e^{-x}}{2} = \cosh x,$$

and, using the quotient rule for derviatives,

$$\frac{\mathrm{d}}{\mathrm{d}x} \tanh x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sinh x}{\cosh x} = \frac{\sinh^2 x - \cosh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x.$$

[1] The inverse function $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ does not have two-sided limits at $y = \pm 1$ so derivatives are not defined at these endpoints (the graph of $\sin^{-1} y$ has asymptotes $y = \pm 1$, the derivative/slope approaches infinity as $y \to \pm 1$). The function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ extends by periodicity to all of $\mathbb{R}$ and is differentiable everywhere with derivative $\cos x$.

(iii) Find an expression for the inverse functions $\sinh^{-1}$, $\cosh^{-1}$ and $\tanh^{-1}$ in terms of the natural logarithm ln. Calculate their derivatives. Let $y = \cosh x = \frac{e^{2x}+1}{2e^x}$. Solving the quadratic $(e^x)^2 - 2ye^x + 1 = 0$ in e^x yields

$$e^x = y \pm \sqrt{y^2 - 1}$$

and since $e^x > 0$ we have, for $y \in (1, \infty)$,

$$\cosh^{-1} y = x = \ln(y + \sqrt{y^2 - 1}).$$

Similarly, solving $y = \sinh x = \frac{e^{2x}-1}{2e^x}$ for e^x yields, for $y \in (-\infty, \infty)$,

$$\sinh^{-1} y = x = \ln(y + \sqrt{y^2 + 1}).$$

Finally, let $y = \tanh x = \frac{e^{2x}-1}{e^{2x}+1}$. Solving the quadratic $(1-y)(e^x)^2 = 1 + y$ that this gives, we have, for $y \in (-1, 1)$,

$$\tanh^{-1} y = x = \ln \sqrt{\frac{1+y}{1-y}} = \frac{1}{2} \ln \frac{1+y}{1-y}.$$

The derivatives of the inverse hyperbolic functions can be calculated in a similar way to the trigonometric functions in question 5(iii), or, since we have explicit expressions for these inverse functions in terms of natural logarithms and $\frac{\mathrm{d}}{\mathrm{d}x} \ln f(x) = \frac{f'(x)}{f(x)}$ for a differentiable function with range contained in $(0, \infty)$, we can use the chain rule to calculate:

$$\frac{\mathrm{d}}{\mathrm{d}y} \cosh^{-1} y = \frac{\mathrm{d}}{\mathrm{d}y} \ln(y + \sqrt{y^2 - 1}) = \frac{1 + y(y^2-1)^{-\frac{1}{2}}}{y + (y^2-1)^{\frac{1}{2}}} = \frac{(y^2-1)^{\frac{1}{2}} + y}{(y^2-1)^{\frac{1}{2}}(y + (y^2-1)^{\frac{1}{2}})} = \frac{1}{\sqrt{y^2-1}},$$

$$\frac{\mathrm{d}}{\mathrm{d}y} \sinh^{-1} y = \frac{\mathrm{d}}{\mathrm{d}y} \ln(y + \sqrt{y^2 + 1}) = \frac{1 + y(y^2+1)^{-\frac{1}{2}}}{y + (y^2+1)^{\frac{1}{2}}} = \frac{(y^2+1)^{\frac{1}{2}} + y}{(y^2+1)^{\frac{1}{2}}(y + (y^2+1)^{\frac{1}{2}})} = \frac{1}{\sqrt{y^2+1}},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y} \tanh^{-1} y = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}[\ln(1+y) - \ln(1-y)] = \frac{1}{2}\left[\frac{1}{1+y} + \frac{1}{1-y}\right] = \frac{1}{(1+y)(1-y)} = \frac{1}{1-y^2}.$$