Mathematical Analysis I

Exercise sheet 9

10 December 2015

References: Abbott 4.5, 4.6, 5.2. Bartle & Sherbert 5.3, 6.1

1. State the Intermediate Value Theorem

(ii) Prove that a polynomial of odd degree has at least one real root.

(iii) Suppose that $f : [a, b] \to \mathbb{R}$ is monotone increasing. Prove that if for each $d \in [f(a), f(b)]$ there is $c \in [a, b]$ such that $f(c) = d$ then f is continuous. [*This gives a converse to the Intermediate Value Theorem for monotone functions.*]

2.

(i) Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that f must have a fixed point. i.e., that $f(c) = c$ for at least one value of $c \in [a, b]$. [*This is a special case of the Brouwer Fixed-Point Theorem.*]

(ii) Show that if a continuous function $f : [0, 1] \to [0, 1]$ has an infinite set of fixed points then not all fixed points are isolated points, i.e., there is a fixed point of f that is a limit point of other fixed points of f.

(iii) Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \cap [0, 1] \\ 1 - x & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$$

Show that f is bijective, that $f(f(x)) = x$ and that f is continuous only at the point $\frac{1}{2}$.

3. Evaluate the following limits:

(i) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

(ii) $\lim_{x \to 0} \frac{\sin x}{x}$. [*You may assume the inequalities* $\sin x < x < \tan x$. *From these you can then derive inequalities that will enable you to apply the Squeeze Theorem to deduce the limiting value of* $\frac{\sin x}{x}$.]

(iii) $\lim_{x \to 0} \frac{e^x - 1}{x}$.

4. For $q \in \mathbb{Q}$, let $g_q : \mathbb{R} \to \mathbb{R}$ be defined by $g_q(x) = x^q \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $g_q(0) = 0$.

(i) Show that the function $g_0(x) = \sin\left(\frac{1}{x}\right)$ is not continuous at 0, and that $g_1(x)$ is continuous at 0 but not differentiable at 0.

(ii) Prove that g_2 is differentiable on $\mathbb{R}$ and calculate $g_2'(x)$. Show that $g_2'(x)$ is discontinuous at 0.

(iii) Find a particular (potentially noninteger) value for q so that

(a) g_q is differentiable on $\mathbb{R}$ but such that g_q' is unbounded on $[0, 1]$.

(b) g_q is differentiable on $\mathbb{R}$ with g_q' continuous but not differentiable at 0.

(c) g_q is differentiable on $\mathbb{R}$ and g_q' is differentiable on $\mathbb{R}$, but such that g_q'' is discontinuous at 0.

5.

(i) A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period T if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Prove that if f is a periodic function that is differentiable everywhere then $f'(x)$ is periodic as well. What is the period of f'?

(ii) Calculate the derivative of the functions $\sin x$ and $\cos x$. Deduce the derivative of $\tan x$ from these. [*For calculating the derivative of* $\sin x$ *use the identity* $\sin(x+h) = \sin x \cos h + \sin h \cos x$, *while for* $\cos x$ *you might use the analogous identity, or use* $\cos x = \sin(x + \frac{\pi}{2})$.]

(iii) Write down domains for the functions sin and cos restricted to which these functions become bijections. Calculate the derivatives of the inverse functions $\sin^{-1}$ and $\cos^{-1}$.

6. The *hyperbolic functions* $\sinh, \cosh, \tanh : \mathbb{R} \to \mathbb{R}$ are defined for $x \in \mathbb{R}$ by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}.$$

(i) Determine the range of each function sinh, cosh and tanh.

(ii) Calculate the derivatives of sinh, cosh and tanh.

(iii) Find an expression for the inverse functions $\sinh^{-1}$, $\cosh^{-1}$ and $\tanh^{-1}$ in terms of the natural logarithm ln. Calculate their derivatives.