Mathematical Analysis I

Exercise sheet 7

Selected solutions

12 November 2015

References: Abbott 2.1, 2.8, 2.9, 3.2

A set $A \subseteq \mathbb{R}$ is open if for every point x of A there is some ϵ-neighbourhood of x contained in A.

A point x is a limit point of A if any neighbourhood of x contains a point of A distinct from x. Equivalently, x is a limit point of A if there is a sequence of points in $A \setminus \{x\}$ convergent to x. (Question 2(ii) and Abbott, Theorem 3.2.5.)

A set $A \subseteq \mathbb{R}$ is closed if it contains all its limit points. (In particular, a finite set A is closed since it has no limit points.)

An isolated point x in a set A is a point that is not a limit point of A: there is a neighbourhood of x that contains no points in A distinct from itself. A finite set consists of isolated points.

4. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true. ($\overline{A}$ denotes the closure of A, i.e., the union of A and its limit points.)

(i) For any set $A \subseteq \mathbb{R}$, the set $\mathbb{R} \setminus \overline{A}$ is open.

The closure of $\overline{A}$ is a closed set (in fact the smallest closed set containing A).

Proof: Let L be the set of limit points of A. Then $\overline{A} = A \cup L$.

To show $\overline{A}$ is closed we prove the equivalent assertion that $\mathbb{R} \setminus \overline{A}$ is open. Let $x \in \mathbb{R} \setminus \overline{A}$. Then x is neither in A nor a limit point of A. Therefore there must be some neighbourhood $V_\epsilon(x)$ of x that does not meet A (otherwise x would be a limit point of A and so belong to L) and which does not meet L (because if it did then by definition of a limit point $V_\epsilon(x)$ would contain another point of A, and we know that $V_\epsilon(x) \cap A = \emptyset$). Hence $V_\epsilon(x)$ is disjoint from A and L, hence from $\overline{A}$, and we conclude that $V_\epsilon(x) \subseteq \mathbb{R} \setminus \overline{A}$. This proves $\mathbb{R} \setminus \overline{A}$ is open, as desired.

(Further, any closed set containing A must contain all its limit points, and hence contains $\overline{A}$.)

The complement $\mathbb{R} \setminus \overline{A}$ of the closed set $\overline{A}$ is an open set (as we have seen in the proof above - or by the general result that complements of closed sets are open and conversely, complements of opens sets are closed – see Abbott, Theorem 3.2.13.)

(ii) If a set A has an isolated point, it cannot be an open set.

If $x \in A$ is an isolated point of A then there is some neighbourhood $V_\epsilon(x)$ of x that contains no other points of A than x. But then there can be no neighbourhood of x contained within A. Hence A is not an open set.

(iii) A set A is closed if and only if $\overline{A} = A$.

From (i) we know that $\overline{A}$ is a closed set, so if $\overline{A} = A$ then A is closed. Conversely, if A is closed then by definition it contains all its limit points, and $\overline{A}$ is the union of A with its limit points and hence equal to A in this case.

(iv) If A is a bounded set, then $s = \sup A$ is a limit point of A. This holds if $s = \sup A$ is not a maximum element of A (i.e. does not belong to A): in this case for every $\epsilon > 0$ there is $x \in A$ such that $s - \epsilon < x < s$. In other words, every ϵ-neighbourhood of s contains a point $x \in A$ distinct from s.

A counterexample is provided by taking A to be a bounded set of isolated points e.g. $A = \{0\}$. Here $\sup A = \max A$ is not a limit point.

(v) Every finite set is closed.

A finite set is a finite union of isolated points $\{x\}$. An isolated point is closed (no limit points to contain). A finite union of closed sets is closed. Hence every finite set is closed.

(vi) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

Given that A is open, for each $x \in \mathbb{A}$ there is some $\epsilon > 0$ such that the neighbourhood $V_\epsilon(x)$ is contained in A. Since $A \supseteq \mathbb{Q}$, by density of $\mathbb{Q}$ in $\mathbb{R}$, for any given $y \in \mathbb{R}$ and $\epsilon > 0$ we can find $x \in \mathbb{Q} \subseteq A$ such that $y \in V_\epsilon(x)$ (there is always a rational x strictly between $y - \epsilon$ and y). Choosing $\epsilon > 0$ such that $V_\epsilon(x) \subseteq A$ this shows that $y \in A$. Hence $\mathbb{R} = A$.