Mathematical Analysis I

Exercise sheet 6

12 November 2015

References: Abbott 2.7. Bartle & Sherbert 3.7

1. Define what it means for an infinite series $\sum_{n=1}^{\infty} a_n$ to converge. Let $s_n = a_1 + a_2 + \cdots + a_n$. The series $\sum_{n=1}^{\infty} a_n$ is said to converge if the sequence (s_n) converges.

   (i) Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $(a_n) \to 0$. $a_n = s_n - s_{n-1}$ for $n \geq 2$ and $\lim a_n = \lim s_n - \lim s_{n-1} = 0$.

   (ii) Give a counterexample to the converse of (i). The converse to (i) states that if $(a_n) \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges. A counterexample is given by $a_n = \frac{1}{n}$ (divergent harmonic series).

   (iii) Let $r \in \mathbb{R}$. Prove that the series $\sum_{n=1}^{\infty} r^n$ converges if and only if $(r^n) \to 0$, and write down its limit in this case. By (i) if the series $\sum_{n=1}^{\infty} r^n$ converges then $(r^n) \to 0$. We need to prove the converse holds in this case. Suppose then that $(r^n) \to 0$, i.e., $|r| < 1$. Let $s_n = r + r^+ \cdots + r^n$. Then $rs_n - s_n = r^{n+1} - r$, whence $s_n = \frac{r - r^{n+1}}{1-r}$, and so

$$s_n - \frac{r}{1-r} = \frac{r^{n+1}}{1-r}.$$

   Hence

$$\left| s_n - \frac{r}{1-r} \right| \leq \frac{|r|^{n+1}}{1-r}.$$

   Since $|r|^{n+1} \to 0$ as $n \to \infty$ when $|r| < 1$, it follows that (s_n) converges to $\frac{r}{1-r}$ as $n \to \infty$ when $|r| < 1$.

2.

   (i) Let (a_n) be a sequence of nonnegative reals. Prove that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence of its partial sums $(a_1 + \cdots + a_n)$ is bounded. The sequence of partial sums $(s_n) = (a_1 + \cdots + a_n)$ is monotone increasing as $a_n \geq 0$ for each n.

   Suppose (s_n) is bounded. Then the Monotone Convergence Theorem implies that (s_n) converges to the limit $\sup\{s_n : n \in \mathbb{N}\}$.

   Conversely, suppose that (s_n) is convergent. Then (s_n) is bounded. (For sufficiently large N the terms of (s_n) with $n \geq N$ lie within 1 of $l = \lim s_n$, and the finitely many terms $s_1, \ldots, s_{N-1}$ are bounded too.)

   (ii) Using the result of (i), prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Set $a_n = \frac{1}{n}$. For $n = 2^k$ we have

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}} + \cdots + \frac{1}{2^{k-1}}\right) \\ &= 1 + \frac{k}{2}. \end{aligned}$$

Thus for $2^k \le n < 2^{k+1}$ we have $s_n > 1 + \frac{k}{2}$, so that (s_n) is unbounded. By (i) we conclude that (s_n) diverges.

(iii) Deduce from (ii) that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \le 1$. Take $a_n = \frac{1}{n^p}$ with $0 \le p < 1$ and (s_n) the sequence of partial sums. Then $a_n \ge \frac{1}{n} > 0$ from which it follows that (s_n) is unbounded, since the same is true of the partial sums of $(\frac{1}{n})$. Therefore $\sum \frac{1}{n^p}$ diverges to $+\infty$ when $0 \le p \le 1$.

(iv) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.
For $n = 3$ we have

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}.$$

For $n = 7$,

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) < 1 + \frac{2}{2^p} + \frac{4}{4^p} = 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}.$$

By induction on k, for $n = 2^k - 1$ we have

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^k - 1)^p} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{k-1}$$

and the right-hand side of this inequality is (by question 1(iii)) bounded above by $\frac{1}{1-\frac{1}{2^{p-1}}}$, so the partial sums (s_n) are bounded, with

$$s_{2^k-1} \le s_n < s_{2^{k+1}-1} < \frac{2^{p-1}}{2^{p-1} - 1}$$

for $2^k - 1 \le n < 2^{k+1} - 1$. By part (i), the series $\sum \frac{1}{n^p}$ converges to a limit $\le \frac{2^{p-1}}{2^{p-1}-1}$ when $p > 1$.

3. Let (a_n) be a sequence of reals that is monotone decreasing and converges to 0.

(i) Prove that the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to a limit between $a_1 - a_2$ and a_1.
Note that $a_n \ge 0$ since $(a_n) \to 0$ and (a_n) is decreasing.
Let $s_n = a_1 - a_2 + \cdots + (-1)^{n+1} a_n$ be the nth partial sum of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. Then, as $a_n \ge a_{n+1}$ for all n,

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n})$$

defines a monotone increasing subsequence (s_{2n}) and

$$s_{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1})$$

defines a monotone decreasing subsequence (s_{2n+1}). Since

$$a_1 - a_2 \le s_{2n} \le s_{2n} + a_{2n+1} = s_{2n+1} \le a_1$$

both these subsequences are bounded below by $a_1 - a_2$ and above by a_1 and converge to the same limit l, and $a_1 - a_2 \le l \le a_1$. It follows that (s_n) converges to this common limit l (since the subsequences (s_{2n}) and (s_{2n+1}) between them contain all the terms of (s_n)).

(ii) Deduce from (i) that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges for $p > 0$. Take $a_n = \frac{1}{n^p}$ in (i), which defines a monotone decreasing series convergent to 0 precisely when $p > 0$: by the Archimedean Property, for any given $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $N > \epsilon^{-\frac{1}{p}}$, or $N^p > \epsilon^{-1}$. Then $\frac{1}{n^p} \le \frac{1}{N^p} < \epsilon$ for $n \ge N$, which shows that $(\frac{1}{n^p}) \to 0$.
[Note that when $p < 0$ the function $x \mapsto x^p$ is no longer monotone increasing, so the inequality $N > \epsilon^{-\frac{1}{p}}$ is switched when raising to the power p in this case and the sequence $(\frac{1}{n^p})$ is no longer monotone decreasing, but increases without bound. When $p = 0$ the series $\sum (-1)^{n+1}$ is divergent as its odd partial sums are constantly 1 and its even partial sums constantly 0. Thus $\sum \frac{(-1)^{n+1}}{n^p}$ diverges when $p \le 0$.]

4. For each of the following series, prove either that it diverges, or that it converges to a limit and in this case determine the limit:

(i) By the ratio test $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges, since $\frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n} \to \frac{1}{2}$ as $n \to \infty$. The partial sum is given for $r = \frac{1}{2}$ by

$$s_n = r + 2r^2 + \cdots + nr^n$$

which satisfies

$$s_n - rs_n = r + r^2 + \cdots + r^n - nr^{n+1} = \frac{r - r^{n+1}}{1-r} - nr^{n+1},$$

(see question 1(iii)) and so

$$s_n = \frac{r - (n+1)r^{n+1} + nr^{n+2}}{(1-r)^2}.$$

The sequence (s_n) thus converges to $\frac{r}{(1-r)^2}$ as $n \to \infty$ (since (r^n) and (nr^n) converge to 0). Taking $r = \frac{1}{2}$ we find that $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{1/4} = 2$.

(ii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges by comparison with $\sum \frac{1}{n^2}$ $(0 < \frac{1}{n(n+1)} < \frac{1}{n^2})$ and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1$$

(iii) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ diverges by the limit comparison test (Bartle & Sherbert Theorem 3.7.8) with the harmonic series $\sum \frac{1}{n}$: the sequence $(n^{\frac{1}{n}})$ converges to 1 (Ex. sheet 5, q. 4(ii)) so that $\lim \frac{1/n\sqrt[n]{n}}{1/n} = 1$. (For sufficiently large n we have $\frac{1}{2n} \leq \frac{1}{n\sqrt[n]{n}} < \frac{1}{n}$ and the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$ to infinity forces the same to be true of $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$.)

(iv) As

$$\frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}$$

we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n} + \frac{1}{n+1}\right) = 1 + \frac{1}{2} - (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{4}) - \cdots = 1,$$

the terms cancelling in pairs, except for the first term 1 (and the terms convergent to 0: $s_{2n+1} = 1$ and $s_{2n} = 1 - \frac{1}{n+1}$, so the whole sequence of partial sums does converge, and not oscillate, as e.g. for $\sum(-1)^n$). The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is the conditionally convergent alternating harmonic series, with limit $\ln 2$. Conditional convergence means rearranging the series will give different results - so for example one might be tempted to write

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} - \sum_{n=1}^{\infty} (-1)^{n+2} \frac{1}{n+1} = \ln 2 - (\ln 2 - 1) = 1.$$

but this requires justification of the rearrangment preserving the limiting value of the series (not easy to do).

(v) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test (see question 3(i)) to a limit between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}$.

5. Let (a_n) be a sequence of strictly positive reals and suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. Either prove or give a counterexample to the following statements:

(i) the series $\sum_{n=1}^{\infty} a_n^2$ converges, If $\sum a_n$ is convergent then (a_n) converges to 0. In particular, $a_n < 1$ for sufficiently large n. Then $a_n^2 < a_n$ for large enough n and by the Comparison Test the series $\sum a_n^2$ also converges.

(ii) the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, Counterexample: take $a_n = \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges but $\sum \frac{1}{n}$ diverges.

(iv) the series $\sum_{n=1}^{\infty} \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$ diverges. We have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{a_1}{n}$$

so by comparison with the unbounded harmonic series $a_1 \sum \frac{1}{n}$ the given series diverges.