Mathematical Analysis I

Exercise sheet 3

22 October 2015

References: Abbott, 1.3, 1.4 and 8.4. Bartle & Sherbert 1.3, 2.3, 2.4, 2.5

1. Define a *Dedekind cut* of the rationals. Fix $r \in \mathbb{Q}$. Show that the set $C_r = \{x \in \mathbb{Q} : x < r\}$ is a Dedekind cut.

   Which of the following subsets of $\mathbb{Q}$ are Dedekind cuts?

   (i) $\{x \in \mathbb{Q} : x^2 < 2\}$

   (ii) $\{x \in \mathbb{Q} : x^3 < 2\}$

   (iii) $\{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$

   (iv) $\{x \in \mathbb{Q} : x^2 \leq 2 \text{ or } x < 0\}$

2.

   (i) Define what it means for a binary relation $\leqslant$ on a set S to be an *ordering*.

   (ii) For Dedekind cuts A and B define the relation "$\leqslant$ " as follows: $A \leqslant B$ if and only if $A \subseteq B$. Show that this defines an ordering on the set of all Dedekind cuts.

3. For $A, B \subseteq \mathbb{R}$ define

$$A + B = \{a + b : a \in A, b \in B\}.$$

   (i) Show that if A and B are Dedekind cuts then so is $A + B$.

   (ii) Let $A = \{x \in \mathbb{Q} : x < a\}$ and $B = \{x \in \mathbb{Q} : x < b\}$ for $a, b \in \mathbb{Q}$. From question 1 we know A and B are Dedekind cuts. What cut is $A + B$?

   (iii) Define the Dedekind cut $O = \{r \in \mathbb{Q} : r < 0\}$. Show that O is an identity for addition of cuts and write down the inverse to a cut A with respect to this operation.

4. Find the suprema and infima of the following sets:

   (i) $\{n \in \mathbb{N} : n^2 < 12\}$

   (ii) $\{\frac{n}{m+n} : m, n \in \mathbb{N}\}$

   (iii) $\{\frac{n}{2n+1} : n \in \mathbb{N}\}$

   (iv) $\{\frac{n}{m} : m + n \leq 12\}$.

5. Let A and B be non-empty subsets of $\mathbb{R}$ which are bounded above. Show that if $A \subseteq B$ then $\sup B \leqslant \sup B$. State and prove the corresponding inequality relating $\inf A$ and $\inf B$ when A and B are bounded below.

6. In this question you may assume that $\mathbb{R}$ is uncountable.

(i) Prove that $(0, \infty)$ is uncountable. [*Two approaches are (a) to argue by contradiction, or (b) to produce a bijection from* $(0, \infty)$ *to* $\mathbb{R}$.]

(ii) Deduce from (i) that the interval $(0, 1)$ of $\mathbb{R}$ is uncountable.

(iii) Every real number in $(0, 1)$ has a non-terminating binary expansion. For example,

$$\frac{1}{2} = 0 \cdot 01111111\ldots$$

is the non-terminating expansion $\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$, while $0 \cdot 1 = \frac{1}{2}$ is terminating, i.e., there are only finitely many 1s.[1] Use this fact and part (ii) to deduce that the set of all infinite subsets of $\mathbb{N}$ is uncountable.

[1] Likewise in decimal, we have $0 \cdot 5 = 0 \cdot 049999\ldots$ and generally every number has a non-terminating expansion (infinitely many non-zero places).