Mathematical Analysis I

Exercise sheet 2

15 October 2015

1. Suppose that a, b, c, d are positive integers such that $\frac{a}{b} < \frac{c}{d}$.

   (i) Prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.
   The fraction $\frac{a+c}{b+d}$ is called the *mediant* of $\frac{a}{b}$ and $\frac{c}{d}$.

   (ii) Prove that for positive rationals u and v the weighted mediant $\frac{ua+vc}{ub+vd}$ also lies between $\frac{a}{b}$ and $\frac{c}{d}$.

   (iii) Assume further in (i) that $bc - ad = 1$. Prove that the mediant of $\frac{a}{b}$ and $\frac{c}{d}$ is the rational with least denominator that lies between them. More precisely, if p, q are positive integers such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ then there are positive integers u, v such that $p = ua + vc$ and $q = ub + vd$. *(You may assume that $\frac{p}{q}$ is a weighted mediant of $\frac{a}{b}$ and $\frac{c}{d}$, i.e., that $p = ua + vc$ and $q = ub + vd$ for some $u, v \in \mathbb{Q}$. Consider the positive differences $\frac{ua+vc}{ub+vd} - \frac{a}{b}$ and $\frac{c}{d} - \frac{ua+vc}{ub+vd}$ along with the given assumption $bc - ad = 1$.)*

2. Suppose a, b are positive integers with $a > b$ such that $|\frac{a}{b} - \sqrt{2}| < \epsilon$. Prove that

   (i)
   $$|\frac{a+2b}{a+b} - \sqrt{2}| < \frac{1}{2+2\sqrt{2}}\epsilon,$$

   (ii)
   $$|\frac{a^2+2b^2}{2ab} - \sqrt{2}| < \frac{1}{2}\epsilon^2.$$

Use (ii) to give a rational approximation to $\sqrt{2}$ that is within 2^{-15} of $\sqrt{2}$.

3. Define what is means for a set to be *countable.*

   (i) Show that the following statements are equivalent:

   (a) S is a countable set.
   (b) There exists a surjection from $\mathbb{N}$ onto S.
   (c) There exists an injection from S into $\mathbb{N}$.

   (ii) Show that the set of all finite subsets of $\mathbb{N}$ is a countable set. Is the set of all subsets of $\mathbb{N}$ countable?

   (iii) Show that the set of all finite sequences of elements from $\mathbb{N}$ is countable. *(The case of sequences of length two is $\mathbb{N} \times \mathbb{N}$. Use this a basis for induction, together with the result, which you may assume, that a countable union of countable sets is again countable.)*

4. A real number is *algebraic* if it is a solution of an equation of the form

$$a_0 + a_1x + a_2x^2 \cdots + a_nx^n = 0, \qquad (1)$$

for some $n \in \mathbb{N}$ and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$.

   (i) Prove that equation (1) has at most n solutions. *(You just need the fact that if polynomial $p(x)$ has root a then $p(x) = (x-a)q(x)$ for some polynomial $q(x)$ of strictly smaller degree.)*

(ii) With the help of results proved in question 3, show that the set of algebraic numbers is countable.

(iii) A real number that is not algebraic is *transcendental*. What can you conclude from (ii) about the size of the set of transcendental numbers?

(iv) Write down what statements you would need to prove to establish that the set of algebraic numbers forms a field. *(You are not asked to prove that the algebraic numbers do indeed form a field: the standard approach is to use* resultants*, which you may wish to look up for enlightenment.)*

(v) Use the number $\sqrt{2}^{\sqrt{2}}$ to prove that there exists irrational numbers a and b such that a^b is rational. *(Hint: suppose the given number is irrational, and raise it to an appropriate power.)*

(The number $\sqrt{2}^{\sqrt{2}}$ is in fact known to be transcendental, although proving this is not easy. A corollary of a theorem of Gelfond and Schneider is that when a is an algebraic number not equal to 0 or 1, and b is an irrational algebraic number then a^b is transcendental.)

6.

(i) *(Archimedean property)* Prove that for any $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $x < n$. Deduce as a corollary that for each $y \in \mathbb{R}$ there is $n \in \mathbb{N}$ with $n - 1 \leq y < n$.

(ii) Show using (i) that for any real number $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for each natural number $n \geq N$.

(iii) *(Density of the rationals)* With the help of (i), prove that for any two real numbers x, y with $x < y$ there is a rational number r with $x < r < y$. Deduce that the set $\{r \in \mathbb{Q} : r^2 < 2\}$ has no least upper bound in $\mathbb{Q}$.

(iv) Deduce from (iii) that the irrationals are also dense in the real line, i.e., for any $x < y \in \mathbb{R}$ there is $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$.

7. The *absolute value* function is defined on $\mathbb{R}$ by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Prove that, for $x, y \in \mathbb{R}$,

(i) $|xy| = |x|\,|y|$

(ii) $|x + y| \leq |x| + |y|$ (when does equality hold?)

(iii) $x^2 \leq y^2$ if and only if $|x| \leq |y|$.

Part (ii) is the *Triangle Inequality* and is often written in the alternative form $|a - b| \leq |a - c| + |c - b|$ for any reals a, b, c (take $x = a - c$ and $y = c - b$).

(iv) Deduce from the Triangle Inequality that, for $x, y \in \mathbb{R}$, $|x - y| \leq |x| + |y|$ and $||x| - |y|| \leq |x - y|$.

(v) Show that $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$ and $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$.

8.

(i) Show that is $S \subseteq \mathbb{R}$ is bounded and $T \subseteq S$ then $\inf S \leq \inf T \leq \sup T \leq \sup S$.

(ii) Suppose $S \subseteq \mathbb{R}$ contains $\sup S$ as an element, i.e., $\sup S = \max S$. Show that if $x \notin S$ then $\sup(S \cup \{x\}) = \sup\{\sup S, x\}$.

(iii) Deduce from (i), using mathematical induction, that any finite set $S \subseteq \mathbb{R}$ contains its supremum, i.e., $\sup S = \max S$ when S is finite.