Mathematical Analysis I

Exercise sheet 11

Solutions

7 January 2016

References: Abbott 6.6. Bartle & Sherbert 6.4

1. Let $f : [a, b] \to \mathbb{R}$ be such that f and its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b).

(i) Let $x_0 \in [a, b]$. Show that the polynomial $P_n(x)$ defined by

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

has the property that $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for each $k = 0, 1, \ldots, n$. [The polynomial P_n is called the *nth Taylor polynomial for f at* x_0.]

The polynomial

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

has kth derivative

$$P_n^{(k)}(x) = \sum_{i=k}^{n} \frac{f^{(i)}(x_0)}{(i-k)!}(x - x_0)^{i-k},$$

since $(x - x_0)^i$ has kth derivative $i(i-1)\cdots(i-k+1)(x-x_0)^{i-k}$, using the chain rule for derivatives and induction on k, and

$$\frac{i(i-1)\cdots(i-k+1)}{i!} = \frac{1}{(i-k)!}.$$

Setting $x = x_0$ gives the result that

$$P_n^{(k)}(x_0) = f^{(k)}(x_0)$$

since $\frac{f^{(i)}(x_0)}{(i-k)!}0^{i-k} = 0$ when $i > k$ and $(x - x_0)^0 = 1$.

(ii) Taylor's Theorem with the Lagrange form for the remainder term states that, for any $x \in [a, b]$ there is $c \in (x_0, x)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

where P_n is the nth Taylor polynomial for f at x_0 defined in (i). Find the Taylor polynomial P_n for e^x at x_0 and show that the remainder term converges to 0 as $n \to \infty$ for each fixed x_0 and x. [*Use the fact that if* (a_n) *is a sequence of positive reals such that* $\lim a_{n+1}/a_n$ *exists and is* < 1 *then* $\lim a_n = 0$.]

Let $f(x) = e^x$. Then $f^{(k)}(x) = e^x$, so that $f^{(k)}(x_0) = e^{x_0}$ and e^x has nth Taylor polynomial at x_0 given by

$$P_n(x) = \sum_{i=0}^{n} \frac{e^{x_0}}{i!}(x - x_0)^i$$

and the remainder term is given by

$$e^x - \sum_{i=0}^{n} \frac{e^{x_0}}{i!}(x-x_0)^i = \frac{e^c}{(n+1)!}(x-x_0)^{n+1},$$

where $c \in (x_0, x)$.

The sequence (a_n) of remainder terms, defined by $a_n = \frac{e^c}{(n+1)!}(x-x_0)^{n+1}$, has the property that

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+2}(x-x_0) \to 0 \qquad \text{as } n \to \infty.$$

Hence $(a_n) \to 0$ as $n \to \infty$ (by the ratio test for sequences given in the hint).

(iii) Find the Taylor polynomial P_n for $f(x) = \sin x$ at $x_0 = 0$ and prove that the remainder term converges to 0 as $n \to \infty$ for each x. We have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$ and $f^{(4)}(x) = \sin x = f(x)$. Thus, for $k \geq 0$,

$$f^{(k)}(x) = \begin{cases} \sin x & k \equiv 0 \pmod 4 \\ \cos x & k \equiv 1 \pmod 4 \\ -\sin x & k \equiv 2 \pmod 4 \\ -\cos x & k \equiv 3 \pmod 4 \end{cases}$$

where $k \equiv r \pmod 4$ means that k leaves remainder r on division by 4. At $x = 0$,

$$f^{(k)}(0) = \begin{cases} 0 & k \equiv 0 \pmod 4 \\ 1 & k \equiv 1 \pmod 4 \\ 0 & k \equiv 2 \pmod 4 \\ -1 & k \equiv 3 \pmod 4 \end{cases}$$

i.e.

$$f^{(k)}(0) = \begin{cases} (-1)^{\frac{k-1}{2}} & k \equiv 1 \pmod 2 \\ 0 & k \equiv 0 \pmod 2 \end{cases}$$

Hence the Taylor polynomial of $f(x) = \sin x$ at $x_0 = 0$ is given by

$$P_{2n-1}(x) = P_{2n}(x) = \sum_{i=0}^{n-1} \frac{(-1)^i}{(2i+1)!} x^i,$$

and the remainder by

$$\sin x - P_{2n-1}(x) = \frac{(-1)^n \sin c}{(2n)!} x^{2n}$$

for odd $2n-1$, for some $c \in (x_0, x)$, and

$$\sin x - P_{2n}(x) = \frac{(-1)^n \cos c}{(2n+1)!} x^{2n+1}$$

for even $2n$, for some $c \in (x_0, x)$.

Using the ratio test for sequences again as in (ii), both these remainder terms converge to 0 as $n \to \infty$. For example, in the first case the ratio of remainders is in absolute value equal to $\frac{|x|}{(2n+1)(2n+2)}$, which converges to 0 as $n \to \infty$, from which we deduce so does the sequence $\frac{(-1)^n \sin c}{(2n)!} x^{2n}$. We conclude that the infinite series

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^i$$

converges pointwise to $\sin x$ (i.e., for each $x \in \mathbb{R}$ the series converges to a limit, and this limit is $\sin x$).

(iv) Find the nth Taylor polynomial for $f(x) = (1+x)^{-m}$ at $x_0 = 0$, where m is a positive integer.

We have $f^{(k)}(x) = (-1)^k m(m+1)\cdots(m+k-1)(1+x)^{-m-k}$ so that the nth Taylor polynomial of $f(x) = (1+x)^{-m}$ at 0 is given by

$$\begin{aligned} P_n(x) &= \sum_{i=0}^{n} \frac{(-1)^i m(m+1)\cdots(m+i-1)}{i!} x^i \\ &= \sum_{i=0}^{n} (-1)^i \binom{m+i-1}{i} x^i. \end{aligned}$$

The Lagrange form of the remainder is

$$(1+x)^{-m} - \sum_{i=0}^{n} (-1)^i \binom{m+i-1}{i} x^i = (-1)^{n+1} \binom{m+n}{n+1} (1+c)^{-m-n-1}$$

for some $c \in (0, x)$.

Exercise: for which values of x does this remainder converge to 0?

2.

(i) Let $f, g : [a, b] \to \mathbb{R}$ be n times differentiable on (a, b). Use induction to prove Leibnitz's rule for the nth derivative of a product

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x),$$

for $x \in (a, b)$. The base case $n = 0$ holds since $\binom{0}{0} = 1$ and $f^{(0)}(x) = f(x), g^{(0)}(x) = g(x)$. The case $n = 1$ is the product rule for derivatives:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Assume the truth of the statement for a given n. Then

$$\begin{aligned} (fg)^{(n+1)}(x) &= [(fg)^{(n)}]' \\ &= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x) \right)' \end{aligned}$$

by induction hypothesis. By linearity of differentiation and applying the product rule for the case $n = 1$,

$$\begin{aligned} (fg)^{(n+1)}(x) &= \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n-k)}(x) g^{(k+1)}(x) \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x) g^{(k)}(x) + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)}(x) g^{(k)}(x) \\ &= f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] f^{(n+1-k)}(x) g^{(k)}(x) + f^{(0)}(x) g^{(n+1)}(x) \\ &= f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x) + f^{(0)}(x) g^{(n+1)}(x) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x), \end{aligned}$$

using Pascal's recurrence for binomial coefficients in the penultimate line. This establishes the inductive step, so that the given formula holds for all $n \geq 0$.

(ii) Let $h(x) = e^{-1/x^2}$ for $x \neq 0$ and $h(0) = 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does *not* converge to 0 as $n \to \infty$ for $x \neq 0$. [*By L'Hospital's Rule,* $\lim_{x\to 0} h(x)/x^k = 0$ *for any* $k \in \mathbb{N}$. *Use (i) to calculate* $h^{(n)}(x)$ *for* $x \neq 0$.]

Let $h(x) = e^{-x^{-2}}$ for $x \neq 0$ and $h(0) = 0$.

First we use L'Hospital's Rule to show that $\lim_{x\to 0} \frac{h(x)}{x^m} = 0$ for any $m \in \mathbb{N}$. Using the ∞/∞ form of L'Hospital's Rule, and the general fact that the derivative of $e^{f(x)}$ is $f'(x)e^{f(x)}$,

$$\begin{aligned}
\lim_{x\to 0} \frac{h(x)}{x^m} &= \lim_{x\to 0} \frac{x^{-m}}{e^{x^{-2}}} \\
&= \lim_{y\to\infty} \frac{y^m}{e^{y^2}} \\
&= \lim_{y\to\infty} \frac{my^{m-1}}{2ye^{y^2}} \\
&= \frac{m}{2} \lim_{y\to\infty} \frac{y^{m-2}}{e^{y^2}} \\
&= \begin{cases} \frac{m(m-2)\cdots 2}{2^{m/2}} \lim_{y\to\infty} \frac{1}{e^{y^2}} & m \text{ even} \\ \frac{m(m-2)\cdots 1}{2^{\frac{m+1}{2}}} \lim_{y\to\infty} \frac{y^{-1}}{e^{y^2}} & m \text{ odd} \end{cases} \\
&= 0
\end{aligned}$$

We then have

$$\begin{aligned}
h'(0) &= \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} \\
&= \lim_{x\to 0} \frac{h(x)}{x} \\
&= 0
\end{aligned}$$

by the above with $m = 1$.

For $x \neq 0$, we have $h'(x) = 2x^{-3}h(x)$ and, by the product rule for derivatives,

$$\begin{aligned}
h''(x) &= -6x^{-4}h(x) + 2x^{-3}h'(x) \\
&= -6x^{-4}h(x) + 4x^{-6}h(x)
\end{aligned}$$

Similarly,

$$\begin{aligned}
h''(0) &= \lim_{x\to 0} \frac{h'(x) - h'(0)}{x - 0} = \lim_{x\to 0} \frac{h'(x)}{x} \\
&= \lim_{x\to 0} \frac{2x^{-3}h(x)}{x} \\
&= \lim_{x\to 0} 2\frac{h(x)}{x^4} = 0
\end{aligned}$$

Setting $f(x) = 2x^{-3}$ and $g(x) = h(x)$ in (i), for $n \geq 1$

$$h^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(n-k+2)!x^{k-n-3}h^{(k)}(x), \tag{1}$$

where we have used the result, easily proved by induction, that

$$(2x^{-3})^{(j)} = (-1)^j(j+2)!x^{-3-j},$$

with $j = n - k$.

Assuming inductively for $0 \leq k \leq n$ that $h^{(k)}(x) = q_k(x^{-1})h(x)$ for some polynomial q_k, which is true for $k = 0, 1, 2$ as we have seen, equation (1) for $h^{(n+1)}(x)$ yields the inductive step showing that $h^{(n+1)}(x) = q_{n+1}(x^{-1})h(x)$ for a polynomial q_{n+1}.

Since

$$h^{(n+1)}(0) = \lim_{x \to 0} \frac{h^{(n)}(x)}{x} = \lim_{x \to 0} x^{-1} q_n(x^{-1})h(x) = 0$$

by the fact that $\lim_{x\to 0} x^{-m}h(x) = 0$ for all $m \in \mathbb{N}$, as proved above, it follows that $h^{(n)}(x) = 0$ for all n.

Thus the Taylor polynomial for $h(x)$ att $x = 0$ is identically zero. The remainder term is then $h(x)$ itself, equal to $h^{(n+1)}(c)/(n + 1)!$ for some $c \in (0, x)$. It cannot converge to 0 as $n \to \infty$ as it is constantly equal to $h(x) = e^{-x^{-2}} \neq 0$.