Mathematical Analysis I

Exercise sheet 1

8 October 2015

1. Express each of the following statements formally in terms of $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, as appropriate, defining the domain of variables and any relational symbols you require. Write too the formulation for the negation of each statement, and, finally, translate this negation into a natural English sentence.

(i) If a number is both even and odd then it is equal to 42.

(ii) Between any two rationals there lies another rational.

(iii) The town barber is a man who shaves all those, and only those, men in town who do not shave themselves.

2.

(i) Prove *De Morgan's laws* for sets X and $A, B \subseteq X$:

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B) \quad \text{and} \quad X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B).$$

Your proof for each should come in two parts, for example in the first showing that $X \backslash (A \cup B) \subseteq (X \backslash A) \cap (X \backslash B)$ *and* $X \backslash (A \cup B) \supseteq (X \backslash A) \cap (X \backslash B)$ *in order to establish the given equality.*

(ii) Let A and B be finite sets. Show that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$. Deduce that $|A| + |B| = |A \cup B| + |A \cap B|$.

3. For a function $f : X \to Y$ and $A \subseteq X$ we define $f(A) = \{f(x) : x \in A\}$. Thus $f(X)$ is the range of f with domain X.

(i) Let $A = \{x \in \mathbb{R} : x \neq 1\}$ and define $f(x) = 2x/(x-1)$ for all $x \in A$.

Prove that f is injective and determine the range of f. Define the inverse function f^{-1}.

(ii) Show that if $f : X \to Y$ and $A, B \subseteq X$ then $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) \subseteq f(A) \cap f(B)$. *[ctd. overleaf]*

(iii) Let $f(x) = x^2$ for $x \in \mathbb{R}$ and $A = \{x \in \mathbb{R} : -1 \leq x \leq 0\}$ and $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Show that $A \cap B = \{0\}$ and $f(A \cap B) = \{0\}$, while $f(A) = f(B) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$. Hence $f(A \cap B)$ is a proper subset of $f(A) \cap f(B)$.

Write down the sets $A \backslash B$ and $f(A) \backslash f(B)$ and show that it is *not* true that $f(A \backslash B) \subseteq f(A) \backslash f(B)$.

4. Which of the following relations on $\mathbb{N}$ are reflexive, which are symmetric, which are transitive?

(i) the relation $a|b$ (read as 'a divides b');

(ii) the relation $a \not| b$ (does not divide);

(iii) for a fixed $m \in \mathbb{N}$, a, b are related if a and b leave the same remainder after division by m.

Now suppose that X is a nonempty set and f a function with domain X. Define a relation on X by declaring x and y to be related if $f(x) = f(y)$. Show this defines an equivalence relation. What are its equivalence classes?

5. Prove that $\sqrt{p}$ is irrational when p is prime. *(You may use the fact that when p is prime, it is the case that if p divides ab then either p divides a or p divides b.)* More generally, can you say for which $n \in \mathbb{N}$ is $\sqrt{n}$ irrational?

6. Two sets A and B are *equinumerous* if there is a bijection $f : A \to B$. Show that the relation of being equinumerous is an equivalence relation.

(i) For $a, b \in \mathbb{R}$ with $a < b$ give an explicit bijection from $A = \{x : a < x < b\}$ onto $B = \{y : 0 < y < 1\}$. Show that $\{x \in \mathbb{R} : x > 0\}$ is equinumerous with $\mathbb{R}$, and, finally, deduce that the set A is equinumerous with $\mathbb{R}$.

(ii) A real number is *algebraic* if it is a solution of an equation of the form

$$a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n = 0,$$

for some $n \in \mathbb{N}$ and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$.

Show that the set of algebraic numbers is equinumerous with $\mathbb{N}$. *(You may assume the fact that a set X is equinumerous with $\mathbb{N}$ if and only if there is a surjection from $\mathbb{N}$ onto X. Start with the fact that $\mathbb{Z}$ is equinumerous with $\mathbb{N}$ and go on to establish that there is a surjection from $\mathbb{N}$ onto the set of algebraic numbers.)*