Linear Algebra I

Reduced row echelon form

Computer package problem

Using Sage or another computer program, compute the row reduced echelon form of each of the following matrices

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6 & 0 & 2 & 0 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 8 & 0 & 2 & 0 & 0 \\ 0 & 7 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 10 & 0 & 2 & 0 & 0 & 0 \\ 0 & 9 & 0 & 3 & 0 & 0 \\ 0 & 0 & 8 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}, \quad \ldots$$

Do you spot any pattern? Do you think it persists?

[Exercise from R. Allenby, *Linear Algebra*, Arnold, 1995]

The reduced row echelon form of these matrices are found to be

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let A_n denote the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 2n-2 & 0 & 2 & \cdots & 0 & 0 \\ \cdots & & \cdots & & \cdots & \\ 0 & \cdots & n+2 & 0 & n-2 & 0 \\ 0 & \cdots & 0 & n+1 & 0 & n-1 \\ 0 & \cdots & 0 & 0 & n & 0 \end{bmatrix}.$$

The sequence above is $A_2, A_3, A_4, A_5, A_6, \ldots$.

It appears that the reduced row echelon form of A_n is the $n \times n$ identity matrix when n is even, and that when $n = 2k+1$ is odd it takes the form

$$\left[\begin{array}{c|c} I_{2k} & \mathbf{x}_k \\ \hline \mathbf{0}^T & 0 \end{array}\right]$$

where I_{2k} is the $2k \times 2k$ identity matrix $\mathbf{x}_k$ is a certain $2k \times 1$ column vector and $\mathbf{0}^T$ the $1 \times 2k$ all-zero row vector.

To guess what $\mathbf{x}_k$ is here, we use Sage to find that A_7, A_9 have reduced row echelon forms

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{20} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{6}{20} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{15}{20} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{70} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{8}{70} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{28}{70} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{56}{70} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Note that we have put the fractions with a common denominator in order to better be able to guess the pattern. We write A_3, A_5 in a similar form

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{6} \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The first thing to guess is the sequence of denominators $2, 6, 20, 70, \ldots$ of the fractions in the last column $\mathbf{x}_k$, for A_{2k+1} with $k = 1, 2, 3, 4, \ldots$.

These appear to be the middle binomial coefficients $\binom{2k}{k}$. (We have $\binom{2}{1} = 2, \binom{4}{2} = 6, \binom{6}{3} = 20, \binom{8}{4} = 70, \ldots$.)

Further, ignoring signs, the numerators of the non-zero fractions in the last column $\mathbf{x}_k$ are the initial k binomial coefficients

$$\binom{2k}{0}, \binom{2k}{1}, \ldots, \binom{2k}{k-1}.$$

The signs of the fractions alternate, finishing with a positive entry.

Thus our conjecture is that

$$\mathbf{x}_k^T = \begin{cases} \frac{1}{\binom{2k}{k}} \left[\begin{array}{ccccccccc} -\binom{2k}{0} & 0 & \binom{2k}{1} & 0 & -\binom{2k}{2} & \cdots & 0 & \binom{2k}{k-1} & 0 \end{array} \right] & k \text{ even} \\ \frac{1}{\binom{2k}{k}} \left[\begin{array}{ccccccccc} \binom{2k}{0} & 0 & -\binom{2k}{1} & 0 & \binom{2k}{2} & \cdots & 0 & \binom{2k}{k-1} & 0 \end{array} \right] & k \text{ odd} \end{cases}$$

More compactly, the ith entry of $\mathbf{x}_k$ is equal to 0 when i is even and $(-1)^{j+k+1}\frac{\binom{2k}{j}}{\binom{2k}{k}}$ when $i = 2j+1$ is odd.

To *prove* that this indeed gives the reduced row echelon form for the matrix A_n is another matter!

Alexander observed that the sum of the entries in the last column is equal to $\frac{1}{2}$. We verify that the sum of the entries in $\mathbf{x}_k$ is equal to

$$\begin{aligned} (-1)^{k+1}\sum_{j=0}^{k-1}\frac{(-1)^j\binom{2k}{j}}{\binom{2k}{k}} &= \frac{1}{2}\frac{(-1)^{k+1}}{\binom{2k}{k}}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j} \quad + \frac{1}{2\binom{2k}{k}}\binom{2k}{k} \\ &= 0 + \frac{1}{2} = \frac{1}{2}, \end{aligned}$$

where we have used the identity $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$ (with $n = 2k$).