Linear Algebra I: 2017/18

Revision Checklist for the Examination

The examination tests knowledge of three definitions, one theorem with its proof (in the written part) and a survey question on one topic (in the oral part). Each definition is followed by the request for illustrative examples or a straightforward problem involving the defined terms. Survey questions involve providing definitions, giving theorem statements, examples and relationships between ideas – proofs for this part are not required beyond the key notions, but you will be asked to recall theorem statements. (In the written part you will be given the statement of a theorem, the task being there to prove it).

The oral part also involves a discussion of your answers to the written part.

The examination consists of at most an hour on the written part and a discussion for up to half an hour. The oral part may come in two parts: as there may be other people taking the examination concurrently, you may be asked to begin with the survey topic discussion before doing the written part, and then return for a short second discussion of the written part once you have completed it. You may need to wait for a period after finishing the written part to be called for the discussion part.

Survey topics

The following gives an indication of likely topics that you may be asked about during the oral part of the examination (be prepared to give definitions, examples, algorithm descriptions, theorems, notable corollaries etc. – proofs will not be asked for in this part).

- elementary row operations and Gaussian elimination
- solving homogeneous and non-homogeneous systems of linear equations
- matrix operations
- invertible and singular matrices
- groups and permutations
- fields, vector spaces and their subspaces
- vector spaces related to a matrix A
- spans, linear independence and bases
- linear transformations, their standard matrices and properties

Written part

The tables below give a selection of the principle definitions, concepts and theorems, and is not exhaustive: knowledge of concepts and facts not mentioned here may occasionally be asked as well. It is also a bit redundant, as, following Poole, we encountered many concepts in the context of Euclidean two- and three-dimensional space and then returned to them in the context of finitely generated vector spaces more generally. All the material can be found in Chapters 1,2,3 and 6 of Poole, except for the Steinitz Exchange Lemma (for which see lecture notes, or e.g. the Wikipedia entry `https://en.wikipedia.org/wiki/Steinitz_exchange_lemma`).

References in the table are to the relevant sections in David Poole, Linear Algebra, A Modern Introduction, 3rd Int. Ed., Brooks Cole, 2011.

See also `http://kam.mff.cuni.cz/~fiala/LA.WT/lse.pdf` for a good overview of most of the topics we covered (some parts we did not cover, such as section 10 on isomorphisms, and in section 4 transpositions of permutations etc.)

Definitions, basic concepts and algorithms

Term, idea, method	Reference
vectors (in $\mathbb{R}^2$ and $\mathbb{R}^3$), position (tail/head), direction	§1.1
column vector, row vector, components	
vector addition and subtraction, scalar multiplication, zero vector	
associativity, commutativity, distributivity	
linear combination and its coefficients	
dot product (of vectors in $\mathbb{R}^n$)	§1.2
length (norm), distance, unit vector, projection	
orthogonal	
vector/parametric equation of a line/plane	§1.3
normal vector to a line/plane, normal form of equation of line/plane	
linear equation for line/plane	
linear equation, coefficients, constant term, solution	§2.1
system of linear equations, solution set, (in)consistent	
homogeneous system of linear equations	
equivalent system of linear equations	
back substitution, coefficient matrix, augmented matrix	
solution to $A\mathbf{x} = \mathbf{b}$ as intersection of hyperplanes (row vectors of A are normal vectors)	§2.3
solution to $A\mathbf{x} = \mathbf{b}$ as coefficients in linear combination of column vectors of A equal to $\mathbf{b}$	
leading entry, pivot, free variable, dependent variable	§2.2
elementary row operation, row equivalent, row reduction, row echelon form	
Gaussian elimination	
rank of a matrix	
pivot, non-pivot	
reduced row echelon form	
Gauss–Jordan elimination	
spanning set, linear independence	
matrix, entries, diagonal entries, square matrix, identity matrix, zero matrix	§3.1
row and column vectors as matrices	
matrix equality, matrix addition and scalar multiplication	
matrix multiplication (product), AB by rows of B, by columns of A, entries as dot products	
transpose, symmetric matrix, skew-symmetric matrix	
inverse, invertible (non-singular) matrix	§3.3
elementary matrix	
Gauss–Jordan computation of inverse matrix	
computation of inverse for matrices over the field $\mathbb{Z}_p$	
factorization of matrix into lower and upper triangular matrices	§3.4
solving $A\mathbf{x} = \mathbf{b}$ for $A = LU$ by solving $L(U\mathbf{x}) = \mathbf{b}$	
(forward and backward substitution)	
permutation matrices	
binary operation, group, Abelian (commutative) group, subgroup	
field, finite field $\mathbb{F}$	
vector space over $\mathbb{F}$, subspace, finitely generated	
row space, column space, null space of a matrix	§3.5
span, spanning set, linearly independent set, basis, dimension	
column rank, row rank, rank of a matrix	
linear transformation. domain, codomain	§3.6, §6.4
standard matrix of a linear transformation (i.e. with respect to standard unit vector basis)	
composition of (linear) transformations, identity transformation, inverse of linear transformation	
range, kernel of linear transformation	§6.5
rank, nullity of linear transformation	
one-to-one, onto, invertible linear transformation	

Propositions/Theorems

The following are worth being familiar with for either the survey topic discussion or the written part. Focus just on the key ideas for proving the statements – at least have a notion why they are true. For more complicated proofs you will be given guidance if needed. In the written part a theorem with a more routine proof will be asked for. Proofs will not be asked for the theorems marked* – they will be perhaps needed to prove something else.

Statement	Reference
Cauchy-Schwarz Inequality,* Triangle Inequality, Pythagoras' Theorem	§1.2
row equivalent iff reducable to same row echelon form	§2.2
number of free variables = number of variables − rank of coefficient matrix	(§2.2 Rank Theorem)
linearly independent column vectors iff homogeneous linear system	§2.3 (Theorem 2.6)
with coefficient matrix these columns has non-trivial solution	
test given vector in span of set of vectors by row reduction of appropriate matrix	
test linear independence by row reduction of appropriate matrix	
m lin. independent row vectors in $\mathbb{R}^n$ iff rank of matrix with these rows is $< m$	§2.3 (Theorem 2.7)
$m > n$ vectors in $\mathbb{R}^n$ must be linearly dependent	§2.3 (Theorem 2.8)
properties of matrix addition, scalar multiplication	§3.2
properties of matrix multiplication, matrix powers, transpose	§3.2 (Theorems 3.3, 3.4)
unique inverse	§3.3 (Theorem 3.6)
properties of inverting matrices, negative matric powers	§3.3 (Theorem 3.9)
equivalent statements to "A is invertible"	(Theorem 3.12)
left inverse implies right inverse (and conversely)	(Theorem 3.13)
uniqueness of LU-factorization	§3.4 (Theorem 3.16)
inverse of permutation matrix = transpose	(Theorem 3.17)
product permutation matrices = permutation matrix of composition	
$\mathbb{Z}_m$ a field iff m prime, Fermat's Little Theorem	
row equivalent matrices have same row space	§3.5 (Theorem 3.20)
null space is a subspace	(Theorem 3.21)
basis for row space, column space, and null space	
two bases for $\mathbb{F}^n$ have the same size	(Theorem 3.23)
row space and column space same dimension (row rank = col rank)	(Theorem 3.24)
Rank-nullity theorem for matrices	(Theorem 3.26)
equivalent statements to $A \in \mathbb{F}^{n\times n}$ is an invertible matrix	(Theorem 3.27)
Steinitz Exchange Lemma*	[not in Poole - see lecture notes]
linearly independent set extends to a basis, two bases have same size	
linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is matrix transformation T_A, $A \in \mathbb{F}^{m\times n}$	(Theorem 3.31)
composition of lin. transfs linear, associative, corresponds to matrix multipln	(Theorem 3.32, 6.16)
inverse of linear transformation is linear	(Theorem 6.24)
linear transformation maps a spanning set to a spanning set of the range	(Theorem 6.15)
range and kernel of linear transformation are subspaces	(Theorem 6.18)
range of T_A is column space of A, kernel is null space	
Rank-nullity theorem for linear transformations	(Theorem 6.19)
linear transformation one-to-one iff kernel is trivial space	(Theorem 6.20)
linear transformation between space of same dimension one-to-one iff onto*	(Theorem 6.21)
a one-to-one lin. transf. maps a lin. ind. set to a lin. ind set*	(Theorem 6.22)
(and hence bases to bases)*	
lin. transf. invertible iff one-to-one and onto*	(Theorem 6.24)