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Polynomials Associated with Nowhere-Zero Flows

Martin Kochol

MÚ SAV, Štefánikova 49, 814 73 Bratislava 1, Slovakia
E-mail: kochol@savba.sk

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In this paper we study relations between nowhere-zero $\mathbb{Z}_k$- and integer-valued flows in graphs and the functions $F_G(k)$ and $I_G(k)$ evaluating the numbers of nowhere-zero $\mathbb{Z}_k$- and k-flows in a graph G, respectively. It is known that $F_G(k)$ is a polynomial for $k>0$. We show that $I_G(k)$ is also a polynomial and that $2^{m(G)}F_G(k) \geqslant I_G(k) \geqslant (m(G)+1)\, F_G(k)$, where $m(G)$ is the rank of the cocycle matroid of G. Finally we prove that $F_G(k+1) \geqslant F_G(k)\cdot k/(k-1)$ and $I_G(k+1) \geqslant I_G(k)\cdot k/(k-1)$ for every $k>1$.

1. INTRODUCTION

A graph admits a nowhere-zero k-flow if its edges can be oriented and assigned values $\pm 1, \ldots, \pm(k-1)$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph. Using nonzero elements of an additive Abelian group A instead of integers, we get a nowhere-zero A-flow in the graph. By Tutte [15, 16], a graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow for any Abelian group A of order k. Nowhere-zero flows in graphs present a concept dual to graph colouring, because, by Tutte [16], a planar graph is k-colourable if and only if its dual admits a nowhere-zero k-flow.

The chromatic polynomial also has a dual concept, namely the flow polynomial introduced by Tutte [16]. If G is a graph, then the value $F_G(k)$ of the flow polynomial of G is equal to the number of nowhere-zero A-flows in G (with respect to arbitrary but fixed orientation of G), whenever A is an Abelian group of order k.

We show that also the number of nowhere-zero k-flows in G can be evaluated by a polynomial $I_G(k)$ for $k>0$. Studying certain equivalence relations on the sets of nowhere-zero k-flows in G and totally cyclic orientations of G we prove that $2^{m(G)}F_G(k) \geqslant I_G(k) \geqslant (m(G)+1)\, F_G(k)$, where $m(G)$ denotes the rank of the cocycle matroid of G. The polynomials $F_G(k)$

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and $I_G(k)$ satisfy $F_G(k+1) \geqslant F_G(k) \cdot k/(k-1)$ and $I_G(k+1) \geqslant I_G(k) \cdot k/(k-1)$ for every integral $k > 1$. These estimates are best possible in general.

For more information about the flow polynomial and related topics see Brylawski and Oxley [3], Tutte [18, 19], and Welsh [20]. More details about nowhere-zero flow problems can be found in Jaeger [6], Seymour [12], Younger [21], Zhang [22], and Kochol [8].

2. NOWHERE-ZERO FLOWS IN GRAPHS

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. We associate with each edge of G two distinct arts, distinct for distinct edges. If one of the arts corresponding to an edge is denoted by x, the other is denoted by x^{-1}. When the ends of an edge e are u and v, one of the arcs corresponding to e is said to be *directed from u to v* (and the other from v to u). In particular, a loop corresponds to two distinct arcs both directed from a vertex to itself. Let $D(G)$ devote the set of arcs on G. Then $|D(G)| = 2\,|E(G)|$.

More formally, following the definition of Bondy and Murty [2], we can define a graph G as a quintuple $(V(G), E(G), f_G, D(G), d_G)$ consisting of a nonempty set of vertices $V(G)$, a set of edges $E(G)$, an incidence function f_G, a set of arcs $D(G) = E(G) \times \{0, 1\}$, and a map $d_G\colon D(G) \to V(G) \times V(G)$. $E(G)$ and $D(G)$ are disjoint from $V(G)$ and the function f_G maps each edge e of G to an unordered pair of (not necessarily distinct) vertices of G, which are called the ends of e. If $d_G(e, 0) = (u, v)$, then $d_G(e, 1) = (v, u)$ and $f_G(e) = uv$. We also say that $(e, 0)$, $(e, 1)$ are arcs on e, $(e, 0)$ is directed from u to v, and $(e, 1)$ is directed from v to u and write $(e, 0) = (e, 1)^{-1}$, $(e, 1) = (e, 0)^{-1}$.

If $X \subseteq D(G)$, then denote by $X^{-1} = \{x \in D(G); x^{-1} \in X\}$. By an *orientation* of G we mean any $X \subseteq D(G)$ such that $X \cup X^{-1} = D(G)$ and $X \cap X^{-1} = \emptyset$. (In other words, an orientation of G can be considered as a directed graph arising from G after endowing each edge by an orientation.) If $W \subseteq V(G)$, then the set of arts of $D(G)$ directed from W to $V(G) \backslash W$ is denoted by $\omega_G^+(W)$. Let $\omega_G^-(W) = (\omega_G^+(W))^{-1}$. We write $\omega_G^+(v)$ and $\omega_G^-(v)$ instead of $\omega_G^+(\{v\})$ and $\omega_G^-(\{v\})$, respectively.

In this paper, every Abelian group is additive. If A is an Abelian group, then an *A-chain* in G is a mapping $\varphi\colon D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. The *support* of φ, denoted by $\sigma(\varphi)$, is the set of edges associated with the arcs of G having nonzero values in φ. An A-chain φ in G is called *nowhere-zero* if $\sigma(\varphi) = E(G)$.

A (nowhere-zero) A-chain φ in G satisfying $\sum_{x \in \omega_G^+(v)} \varphi(x) = 0$ for every vertex v of G is called a, *(nowhere-zero) A-flow* in G. If k is a positive integer, then by a *(nowhere-zero) k-flow* φ in G we mean a (nowhere-zero) $\mathbb{Z}$-flow in G such that $|\varphi(x)| < k$ for every $x \in D(G)$. This concept coincides with the usual definition of nowhere-zero flows in graphs presented, e.g., in Jaeger [6] (and also mentioned in the introduction). Note that a graph G has a nowhere-zero 1-flow if and only if $E(G) = \emptyset$. It is well-known that a graph with a bridge does not admit a nowhere zero A-flow for any Abelian group A.

Let $m(G) = |E(G)| - |V(G)| + c(G)$ where $c(G)$ denotes the number of components of G. $m(G)$ is the number of edges obtained after deleting the edges of a spanning forest from G (in [1], $m(G)$ is called the cyclomatic number of G).

By Tutte [16] (see also [6, 22]), for every graph G, there exists a polynomial function $F_G(k)$ of k, called the *flow polynomial* on G, such that the value $F_G(k)$ is equal to the number of nowhere-zero A-flows in G, whenever A is an Abelian group of order k. Note that $F_G(k) = 1$ if $E(G) = \emptyset$ (there is exactly one mapping $\varphi\colon \emptyset \to A$, namely the empty set) and $F_G(k) = 0$ if G has a bridge. In all other cases $F_G(k)$ has degree $m(G)$.

Thus the number of nowhere-zero A-flows in a graph does not depend on the structure of A but only on its order. In this paper we study only $\mathbb{Z}_k$- and k-flows in graphs.

3. EQUIVALENCES AND ORIENTATIONS ASSOCIATED WITH FLOWS

If φ is a nowhere-zero k-flow in a graph G, then denote by $[\varphi]_k$ the nowhere-zero $\mathbb{Z}_k$-flow in G such that $[\varphi]_k(x) = [\varphi(x)]_k$ for every $x \in D(G)$. Define an equivalence relation $\theta_{G,k}$ on the set of nowhere-zero k-flows in G so that nowhere-zero k-flows φ and φ' are $\theta_{G,k}$-equivalent if $[\varphi]_k = [\varphi']_k$. Let $[\varphi]\,\theta_{G,k}$ denote the $\theta_{G,k}$-class containing φ (the set of nowhere-zero k-flows in G which are $\theta_{G,k}$-equivalent to φ). By Tutte [15], if ψ is a nowhere-zero $\mathbb{Z}_k$-flow in G, then there exists a nowhere-zero k-flow φ in G such that $[\varphi]_k = \psi$ (see also [22]). Thus, by Tutte [15], the following holds.

LEMMA 1. *Let G be a graph. Then the mapping $[\varphi]\,\theta_{G,k} \mapsto [\varphi]_k$ is a bijection from the set of $\theta_{G,k}$-classes to the set of nowhere-zero $\mathbb{Z}_k$-flows in G.*

A *circuit* is a connected graph with all vertices of valency two. A *cycle* is a graph with all vertices of even valency. A *directed circuit (directed-cycle)* is an orientation of a circuit (cycle) such that for every vertex v, the number of arts entering v equals the number of arcs leaving v. (Note that $\emptyset \subseteq D(G)$ is also a directed cycle.) If C is a directed cycle in $D(G)$, then denote by φ_C

the 2-flow in G such that $\varphi_C(x) = 1$ for $x \in C$, $\varphi_C(x) = -1$ for $x \in C^{-1}$, and $\varphi_C(x) = 0$ otherwise.

Let φ be a nowhere-zero $\mathbb{Z}$-chain in G. Then $\{x \in D(G); \varphi(x) > 0\}$ is an orientation of G. It is denoted by X_φ and called the *positive orientation of* φ. The following statement is well known (see [5, 11, 22]).

LEMMA 2. *An orientation X of a connected graph G is a positive orientation of a nowhere-zero k-flow if and only if $|\omega_G^-(W) \cap X| / |\omega_G^+(W) \cap X| \leqslant k-1$ for every $\emptyset \subset W \subset V(G)$.*

COROLLARY 1. *Suppose X is an orientation in a graph G. Then the following conditions are pairwise equivalent.*

(a) *X is a positive orientation of a nowhere-zero $\mathbb{Z}$-flow in G.*

(b) *For every two distinct vertices u, v from one component of G there exist directed paths from u to v and from v to u.*

(c) *Every arc of X is covered by a directed circuit in X.*

Proof. (a) $\Rightarrow$ (b) follows from Lemma 2 and the max-flow min-cut theorem. Implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are easy to check. ∎

LEMMA 3. *If φ and φ' are $\theta_{G,k}$-equivalent nowhere-zero k-flows in a graph G, then $X_\varphi \setminus X_{\varphi'}$ is a directed cycle. On the other hand, if C is a directed cycle in X_φ, then these exists precisely one $\varphi'' \in [\varphi]\,\theta_{G,k}$, namely $\varphi'' = \varphi - k\varphi_C$, such that $C = X_\varphi \setminus X_{\varphi''}$.*

Proof. Clearly $\varphi'(x)$ is either $\varphi(x) - k$ or $\varphi(x)$ for every $x \in X_\varphi$. Thus $\tilde{\varphi} = (\varphi - \varphi')/k$ is a 2-flow in G such that $\tilde{\varphi}(x) = 1$ iff $x \in X_\varphi \setminus X_{\varphi'}$. Every 2-flow in G equals $\varphi_{C'}$ for some directed cycle C', whence $X_\varphi \setminus X_{\varphi'}$ is a directed cycle. Conversely, if C is a directed cycle in X_φ, then $\varphi'' = \varphi - k\varphi_C$ is $\theta_{G,k}$-equivalent to φ and $C = X_\varphi \setminus X_{\varphi''}$. Furthermore, if $\bar{\varphi}$ is $\theta_{G,k}$-equivalent to φ'' and $X_{\bar{\varphi}} = X_{\varphi''}$, then $\bar{\varphi} = \varphi''$. ∎

An orientation X of a graph in which every arc is covered by a directed circuit in X is called *totally cyclic*. Let $\mathscr{C}(G)$ denote the set of all totally cyclic orientations of G. Let Θ_G be a relation on $\mathscr{C}(G)$ such that $(X, X') \in \Theta_G$ if $X \setminus X'$ is a directed cycle, $X, X' \in \mathscr{C}(G)$.

LEMMA 4. *Θ_G is an equivalence relation on $\mathscr{C}(G)$.*

Proof. By Lemma 2, if $k \geqslant |E(G)|$, then every $X \in \mathscr{C}(G)$ has a nowhere-zero k-flow satisfying $X_\varphi = X$. Furthermore, by Lemma 3, $(\varphi_1, \varphi_2) \mapsto (X_{\varphi_1}, X_{\varphi_2})$ is a surjection from the set of pairs of $\theta_{G,k}$ (formally, an equivalence is a set of pairs) to the set of pairs of Θ_G. Therefore Θ_G is an equivalence, since $\theta_{G,k}$ is so. ∎

COROLLARY 2. *Let G be a graph and φ, φ' be nowhere-zero k-flows in G, $k>0$. Then*

(a) *if φ is $\theta_{G,k}$-equivalent to φ', then X_φ is Θ_G-equivalent to $X_{\varphi'}$;*

(b) *the number of directed cycles in X_φ is equal to $|[\varphi]\,\theta_{G,k}| = |[X_\varphi]\,\Theta_G|$.*

Proof. Follows directly from Lemma 3. ∎

EXAMPLE 1. Let H_n be a graph consisting of two vertices v_1, v_2 and n parallel edges $e_1, ..., e_n$, $n \geqslant 2$. Let $x_1, ..., x_n$ be the arcs on $e_1, ..., e_n$ directed from v_1 to v_2, respectively. Consider a nowhere-zero n-flow φ_n in H_n such that $\varphi_n(x_1) = n-1$ and $\varphi_n(x_2) = \varphi_n(x_3) = \cdots = \varphi_n(x_n) = -1$. Then $X_{\varphi_n} = \{x_1, x_2^{-1}, x_3^{-1}, ..., x_n^{-1}\}$, and, by Corollary 2(b), $|[\varphi_n]\,\theta_{H_n,k}| = |[X_{\varphi_n}]\,\Theta_{H_n}| = n$ for every $k \geqslant n$.

Consider the graph H_4 and nowhere-zero 3-flows φ and φ' in H_4 such that $2\varphi(x_1) = \varphi(x_2) = -2\varphi(x_3) = -\varphi(x_4) = 2$ and $\varphi'(x_1) = 2\varphi'(x_2) = -\varphi'(x_3) = -2\varphi'(x_4) = 2$. Then $\varphi \not\equiv \varphi'(\theta_{H_4,k})$ for every $k \geqslant 3$. On the other hand $X_\varphi = X_{\varphi'} = \{x_1, x_2, x_3^{-1}, x_4^{-1}\}$, and, by Corollary 2(b), $|[\varphi]\,\theta_{H_4,k}| = |[\varphi']\,\theta_{H_4,k}| = |[X_\varphi]\,\Theta_{H_4}| = 6$ for every $k \geqslant 3$.

Let $I_G(k)$ denote the number of nowhere-zero k-flows in a graph G. For every orientation X of G, let $I_X(k)$ denote the number of nowhere-zero k-flows φ in G such that $X_\varphi = X$. By Corollary 1, $I_X(k) = 0$ if X is not totally cyclic.

Denote by $\mathscr{C}_\Theta(G)$ the set of equivalence classes of Θ_G. $\mathscr{C}_\Theta(G)$ is a partition of $\mathscr{C}(G)$ (the set of totally cyclic orientations of G). If $\mathscr{X} \in \mathscr{C}_\Theta(G)$ and $X, X' \in \mathscr{X}$, then, by Lemma 3, $\varphi \mapsto \varphi - k\varphi_{X \setminus X'}$ is a bijection between the sets of nowhere-zero k-flows in G with positive orientations X and X', respectively. Thus $I_X(k) = I_{X'}(k)$ for every $k>0$. Denote this value by $I_{\mathscr{X}}(k)$. We have

$$I_G(k) = \sum_{X \in \mathscr{C}(G)} I_X(k) = \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k) \cdot |\mathscr{X}| \qquad (k>0). \tag{1}$$

Consider $\mathscr{X} \in \mathscr{C}_\Theta(G)$ and $k>0$. Let T be the set of nowhere-zero k-flows φ in G such that $X_\varphi \in \mathscr{X}$. Then $|T| = \sum_{X \in \mathscr{X}} I_X(k) = |\mathscr{X}| \cdot I_{\mathscr{X}}(k)$. By Corollary 2, if $\varphi \in T$, then $[\varphi]\,\theta_{G,k} \subseteq T$ and, furthermore, $|[\varphi]\,\theta_{G,k}| = |[X_\varphi]\,\Theta_G| = |\mathscr{X}|$. Thus T can be partitioned into $I_{\mathscr{X}}(k)$ sets which are $\theta_{G,k}$-classes. Since, by Lemma 1, $F_G(k)$ is the number of all $\theta_{G,k}$-classes, we get the following formula.

$$F_G(k) = \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k) = \sum_{X \in \mathscr{C}(G)} \frac{I_X(k)}{|[X]\,\Theta_G|} \qquad (k>0). \tag{2}$$

4. POLYNOMIALS $I_G(k)$ AND $I_X(k)$

We expect familiarity with basic properties of polytopes (see, e.g., [10, 13, 23]). Every polytope P in $\mathbb{R}^n$ is the convex hull of the (finite) set of its vertices. If these are integral, then P is called *integral*. The dimension of a polytope is the dimension of its affine hull. By an *extended interior* of a polytope P we mean any nonempty set arising from P after deleting some faces of P. Every extended interior of a polytope is a convex set. The following statement is proved by Ehrhart [4] (see also Stanley [14]).

LEMMA 5. *If $\bar{P}$ is an extended interior of an integral polytope P, then these exists a polynomial f such that the number of integral vectors from $k\bar{P}$ is equal to $f(k)$ for every $k > 0$. The degree of f is equal to the dimension of P.*

THEOREM 1. *Let X be an orientation of a graph G. Then there exists a polynomial function f of k such that $I_X(k) = f(k)$. The degree of f is $m(G)$ if X is totally cyclic and $f = 0$ otherwise.*

Proof. As already noted if X is not totally cyclic, then $I_X(k) = 0$. Assume that X is totally cyclic, and let $U_X(\bar{U}_X)$ be the set of mappings $\varphi\colon X \to \mathbb{R}$ which can be extended to $\mathbb{R}$-flows in G and satisfy $0 \leqslant \varphi(x) \leqslant 1 (0 < \varphi(x) < 1)$ for every $x \in X$. Considering U_X and $\bar{U}_X$ as vectors indexed by X, we get that $\bar{U}_X$ is an extended interior of U_X and $I_X(k)$ equals the number of integral points in $k\bar{U}_X$. By Lemma 5, it suffices to show that U_X is an integral polytope of dimension $m(G)$. This follows from a result of Tutte [17] (see also [13, Chapter 19.3]) who proved that $U_X = \{\mathbf{x} \in \mathbb{R}^X;\, M\mathbf{x} \leqslant \mathbf{b}\}$ where M is a totally unimodular matrix of rank $|V(G)| - c(G)$ and $\mathbf{b}$ is an integral vector. Thus $I_X(k)$ is a polynomial of degree $m(G)$ (see also Remark 2 below). ∎

THEOREM 2. *Let G be a graph. Then there exists a polynomial function f of k such that $F_G(k) = f(k)$. The degree of f is $m(G)$ if G is bridgeless and $f = 0$ otherwise.*

Proof. It is known that every bridgeless graph has a totally cyclic orientation. Thus the statement follows from Theorem 1 and (1). ∎

We call $I_G(k)$ and $I_X(k)$ the *integral flow polynomials* of G and X, respectively.

5. BOUNDS BETWEEN $F_G(k)$ AND $I_G(k)$

The following statement is proved in [9].

LEMMA 6. *Let X be a totally cyclic orientation of a graph G. Then there exists an $m(G)$-tuple $(C_1, ..., C_{m(G)})$ of directed circuits in X covering all arcs of X with the property that every C_i contains an arc $x_i (i = 1, ..., m(G))$ such that $x_i \notin C_j$ for each $j < i$.*

In order to obtain an upper bound for the number of directed cycles in an orientation of a graph we will introduce one more concept. Let X be an orientation of a graph G and $\mathbf{u}$ be an integer-valued function on $V(G)$. Then $Y \subseteq X$ is called a $\mathbf{u}$-*suborientation* of X if $|\omega_G^+(v) \cap Y| - |\omega_G^-(v) \cap Y| = \mathbf{u}(v)$ for every vertex v of G. Clearly, directed cycles in X are precisely the $\mathbf{0}$-suborientations of X where $\mathbf{0}(v) = 0$ for every $v \in V(G)$.

LEMMA 7. *If X is an orientation of a graph G and $\mathbf{u}$ is an integer-valued function on $V(G)$, then the number of $\mathbf{u}$-suborientations of X is at most $2^{m(G)}$.*

Proof. We use induction on $m(G)$. If $m(G) = 0$, then G is a forest and we can easily check that the number of $\mathbf{u}$-suborientations of X is at most $1 = 2^{m(G)}$. If $m(G) > 0$, then G has an edge e which is not a bridge. Thus $m(G-e) = m(G) - 1$. Let x be the arc from X which corresponds to e. Suppose x is directed from v_1 to v_2 . Take $\mathbf{u}': V(G) \to \mathbb{Z}$ so that $\mathbf{u}' = \mathbf{u}$ if $v_1 = v_2$ and otherwise $\mathbf{u}'(v_1) = \mathbf{u}(v_1) - 1$, $\mathbf{u}'(v_2) = \mathbf{u}(v_2) + 1$, and $\mathbf{u}'(v) = \mathbf{u}(v)$ for every $v \in V(G) \setminus \{v_1, v_2\}$. Let $X' = X \setminus \{x\}$. Since X' is an orientation of $G-e$, then, by the induction hypothesis, there are at most $2^{m(G-e)}$ $\mathbf{u}$-suborientations ($\mathbf{u}'$-suborientations) of X'. Thus there are at most $2 \cdot 2^{m(G-e)} = 2^{m(G)}$ $\mathbf{u}$-suborientations of X. ▮

COROLLARY 3. *Every totally cyclic orientation of a graph G has at least $m(G) + 1$ and at most $2^{m(G)}$ directed cycles.*

Proof. Follows from Lemmas 6, 7, and the fact that the directed cycles in an orientation X of G are the $\mathbf{0}$-suborientations in X. (Note that $\varnothing$ is among the directed cycles.) ▮

THEOREM 3. *Let G be a graph. Then $2^{m(G)} F_G(k) \geqslant I_G(k) \geqslant (m(G) + 1)\, F_G(k)$ for every $k > 0$.*

Proof. By Corollaries 2(b) and 3, $2^{m(G)} \geqslant |\mathscr{X}| \geqslant m(G) + 1$ for every $\mathscr{X} \in \mathscr{C}_\Theta(G)$. Then, by (1) and (2), $2^{m(G)} F_G(k) = \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k)\, 2^{m(G)} \geqslant \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k)\, |\mathscr{X}| = I_G(k) \geqslant \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k)\, (m(G) + 1) = (m(G) + 1)\, F_G(k)$. ▮

Remark 1. Let G be a union of two edge-disjoint subgraphs H and H' with at most one vertex in common. Then $F_G(k) = F_H(k) \cdot F_{H'}(k)$ and $I_G(k) = I_H(k) \cdot I_{H'}(k)$.

EXAMPLE 2. If G is a forest, then $m(G)=0$ and $2^{m(G)}F_G(k)=I_G(k)=(m(G)+1)\,F_G(k)$ for every $k>0$. We also give less trivial examples which show that Corollary 3 and Theorem 3 are best possible in a certain sense.

Let $X=\{x_1, x_2^{-1}, x_3^{-1}, ..., x_n^{-1}\}$ be the orientation of H_n described in Example 1. Then X has precisely $n-1$ directed circuits and $n=m(H_n)+1$ directed cycles. Moreover if $n=2, 3$, then every totally cyclic orientation of H_n has precisely n directed cycles. Therefore, if G is homeomorphic with H_2 or H_3, then $I_G(k)=nF_G(k)=(m(G)+1)\,F_G(k)$.

Let G be a graph consisting of n blocks $G_1, ..., G_n$, which are circuits (or, equivalently, which are homeomorphic with H_2). Then $m(G)=n$ and $I_{G_i}(k)=2F_{G_i}(k)$ for $i=1, ..., n$. By Remark 1, $I_G(k)=2^nF_G(k)=2^{m(G)}F_G(k)$.

Remark 2. The degree of $I_X(k)$ can be established without using dimension of polytopes. Let X be a totally cyclic orientation of a graph G. By (2), $F_G(k)\geqslant I_X(k)$; thus $I_X(k)$ can have degree at most $m(G)$. Let $(C_1, ..., C_{m(G)})$ and $(x_1, ..., x_{m(G)})$ be $m(G)$-tuples satisfying the assumptions from Lemma 6. Then for every positive integer s and an $m(G)$-tuple of integers $\mathbf{a}=(a_1, ..., a_{m(G)})$ satisfying $1\leqslant a_i\leqslant s$ we get a nowhere-zero $(m(G)\cdot s+1)$-flow $\varphi_{\mathbf{a}}=\sum_{i=1}^{m(G)} a_i\varphi_{C_i}$ with positive orientation X. If $\mathbf{a}=(a_1, ..., a_{m(G)})\neq\mathbf{a}'=(a'_1, ..., a'_{m(G)})$ and j is the largest index satisfying $a_j\neq a'_j$, then $\varphi_{\mathbf{a}}(x_j)\neq\varphi_{\mathbf{a}'}(x_j)$. Therefore, $\mathbf{a}\mapsto\varphi_{\mathbf{a}}$ is an injective mapping. Thus $I_X(m(G)\cdot s+1)\geqslant s^{m(G)}$, which implies that $\liminf_{k\to\infty} I_X(k)/k^{m(G)}>0$, whence $I_X(k)$ has degree at least $m(G)$.

6. GROWTH OF FLOW POLYNOMIALS

THEOREM 4. *Let G be a bridgeless graph, $E(G)\neq\varnothing$, and let X be a totally cyclic orientation of G. Then for every $k>1$,*

$$I_X(k+1)\geqslant I_X(k)\cdot k/(k-1),$$

$$F_G(k+1)\geqslant F_G(k)\cdot k/(k-1),$$

$$I_G(k+1)\geqslant I_G(k)\cdot k/(k-1),$$

Proof. By (1) and (2), it suffices to prove the statement for $I_X(k)$. Suppose that $I_X(k)>0$ and let $S(T)$ denote the set of nowhere-zero k-flows ($(k+1)$-flows) in G with positive orientation X. For every $\varphi\in S$ and any nonempty directed cycle C in X, define by C-*lift* (simply a *lift*) of φ the unique flow in $T\setminus S$ of the form $\varphi+r\varphi_C$ where r is a positive integer. Let n be the number of nonempty directed cycles in X. Then there exists exactly $n\,|S|$ lifts of φ. Each such lift can be obtained at most $n(k-1)$ times.

Thus $|T \setminus S| \geqslant |S|/(k-1)$, whence $I_X(k+1) = |T| \geqslant |S| \cdot k/(k-1) = I_X(k) \cdot k/(k-1)$. ∎

COROLLARY 4. *Let X be a totally cyclic orientation of a bridgeless graph G, $E(G) \neq \emptyset$, and assume $I_X(k+1)$, $F_G(k+1)$, $I_G(k+1) > 0$. Then*

$$I_X(k+1) \geqslant I_X(k) + 1,$$

$$F_G(k+1) \geqslant F_G(k) + 1,$$

$$I_G(k+1) \geqslant I_G(k) + m(G) + 1.$$

Proof. If $I_X(k+1) > 0$, then in the proof of Theorem 4 it follows that $|T| > |S|$ and $I_X(k+1) \geqslant I_X(k) + 1$. Since $F_G(k+1), I_G(k+1) > 0$, there exists, by (1), (2), $\mathscr{X} \in \mathscr{C}_\Theta(G)$ such that $I_{\mathscr{X}}(k) > 0$; thus also $I_{\mathscr{X}}(k+1) \geqslant I_{\mathscr{X}}(k) + 1$. Hence, by (2), $F_G(k+1) \geqslant F_G(k) + 1$. By Corollary 3, $|\mathscr{X}| \geqslant m(G) + 1$ for every $\mathscr{X} \in \mathscr{C}_\Theta(G)$, whence, by (1), $I_G(k+1) = \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k+1) |\mathscr{X}| \geqslant \sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} (I_{\mathscr{X}}(k)+1) |\mathscr{X}| \geqslant (\sum_{\mathscr{X} \in \mathscr{C}_\Theta(G)} I_{\mathscr{X}}(k) |\mathscr{X}|) + m(G) + 1 = I_G(k) + m(G) + 1$. ∎

Let C_n be the circuit of length n. Then $m(C_n) = 1$ and $F_{C_n}(k) = k-1$. C_n has exactly two totally cyclic orientations X_1, X_2 and $I_{X_1}(k) = I_{X_2}(k) = k-1$. By (1), $I_{C_n}(k) = 2(k-1)$. Thus the bounds from Theorem 4 and Corollary 4 are best possible in this case.

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REFERENCES

1. C. Berge, "Graphs," 2nd ed., North-Holland, Amsterdam, 1985.
2. J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications," Macmillan, London, 1976.
3. T. Brylawski and J. Oxley, The Tutte polynomial and its applications, *in* "Matroid Applications" (N. White, Ed.), pp. 123–225, Cambridge Univ. Press, Cambridge, UK, 1992.
4. E. Ehrhart, Sur un problème de géométrie diophantienne linéaire I, II, *J. Reine Angew. Math.* **226** (1967), 1–29; **227** (1967), 25–49.
5. A. J. Hoffman, Some recent applications of the theory of linear inequalities to extremal combinatorial analysis, *in* "Combinatorial Analysis" (R. Bellman and M. Hall Jr., Eds.), Proc. Sympos. Appl. Math., Vol. 10, pp. 113–127, Amer. Math. Soc., Providence, RI, 1960.
6. F. Jaeger, Nowhere-zero flow problems, *in* "Selected Topics in Graph Theory 3" (L. W. Beineke and R. J. Wilson, Eds.), pp. 71–95, Academic Press, New York, 1988.

7. M. Kochol, Hypothetical complexity of the nowhere-zero 5-flow problem, *J. Graph Theory* **28** (1998), 1–11.
8. M. Kochol, Superposition and constructions of graphs without nowhere-zero k-flows, *European J. Combin.*, to appear.
9. M. Kochol, On bases of the cycle and cut spaces in digraphs, *Ars Combin.*, to appear.
10. V. Klee and P. Kleinschmidt, Convex polytopes and related complexes, *in* "Handbook of Combinatorics" (R. L. Graham, M. Grötschel, and L. Lovász, Eds.), pp. 875–917, North-Holland, Amsterdam, 1995.
11. G. J. Minty, A theorem on three-colloring the edges of a trivalent graph, *J. Combin. Theory* **2** (1967), 164–167.
12. P. D. Seymour, Nowhere-zero flows, *in* "Handbook of Combinatorics" (R. L. Graham, M. Grötschel, and L. Lovász, Eds.), pp. 289–299, North-Holland, Amsterdam, 1995.
13. A. Schrijver, "Theory of Linear and Integer Programming," Wiley, Chichester, 1986.
14. R. P. Stanley, "Enumerative Combinatorics," Vol. 1, Wadsworth and Brooks/Cole, Monterey, 1986.
15. W. T. Tutte, On the imbedding of linear graphs in surfaces, *Proc. London Math. Soc. Ser. 2* **51** (1950), 474–483.
16. W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6** (1954), 80–91.
17. W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 1–47.
18. W. T. Tutte, The dichromatic polynomial, *in* "Proc. Fifth British Comb. Conf. 1975" (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), Congr. Numer. 15, pp. 605–635, Utilitas Math., Winnipeg, 1976.
19. W. T. Tutte, "Graph Theory," Cambridge Univ. Press, Cambridge, UK, 1984.
20. D. J. A. Welsh, "Complexity: Knots, Colourings and Counting," Cambridge Univ. Press, Cambridge, UK, 1993.
21. D. H. Younger, Integer flows, *J. Graph Theory* **7** (1983), 349–357.
22. C.-Q. Zhang, "Integral Flows and Cycle Covers of Graphs," Dekker, New York, 1997.
23. G. M. Ziegler, "Lectures on Polytopes," Springer-Verlag, Berlin, 1995.