Graph polynomials from simple graph sequences

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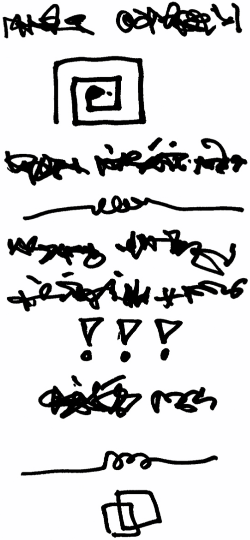
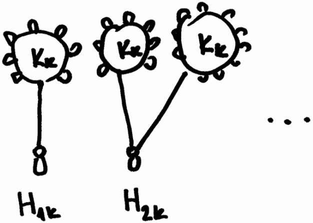



Kk
Kk
Kk
H1k
H2k
...

Chromatic polynomial

Definition by evaluations at positive integers

$k \in \mathbb{N}, \quad P(G;k) = \#\{\text{proper vertex } k\text{-colourings of } G\}.$

$$P(G;k) = \sum_{1 \le j \le |V(G)|} a_j(G) k^{\underline{j}}$$

$$a_j(G) = \#\{\text{partitions of } V(G) \text{ into } j \text{ independent subsets}\},$$

$$P(G;k) = \sum_{1 \le j \le |V(G)|} (-1)^j b_j(G) k^{|V(G)|-j}$$

$$b_j(G) = \#\{j\text{-subsets of } E(G) \text{ containing no broken cycle}\}.$$

$uv \in E(G), \quad P(G;k) = P(G \backslash uv; k) - P(G/uv; k)$

Independence polynomial

Definition by coefficients

$$I(G;x) = \sum_{1 \le j \le |V(G)|} b_j(G) x^j,$$

$$b_j(G) = \#\{\text{independent subsets of } V(G) \text{ of size } j\}.$$

$$v \in V(G), \qquad I(G;x) = I(G-v;x) + xI(G-N[v];x)$$

$I(L(G);x) =$ matching polynomial of G

(Chudnovsky & Seymour, 2006) $K_{1,3} \not\subseteq_i G \Rightarrow I(G;x)$ real roots

$$b_j^2 \ge b_{j-1} b_{j+1}, \qquad \text{(implies } b_1, \ldots, b_{|V(G)|} \text{ unimodal)}$$

Definition

Graphs G, H.
$f : V(G) \to V(H)$ is a *homomorphism* from G to H if $uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$.

Definition

H with adjacency matrix $(a_{s,t})$, weight $a_{s,t}$ on $st \in E(H)$,

$$\hom(G,H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

$$\begin{aligned}\hom(G,H) &= \#\{\text{homomorphisms from } G \text{ to } H\} \\ &= \#\{H\text{-colourings of } G\}\end{aligned}$$

when H simple $(a_{s,t} \in \{0,1\})$ or multigraph $(a_{s,t} \in \mathbb{N})$

The main question

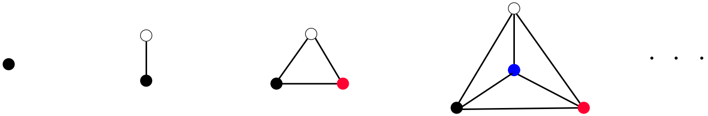
Which sequences $(H_{k,\ell,\dots})$ of simple graphs are such that, for all graphs G, for each $k, \ell, \dots \in \mathbb{N}$ we have

$$\hom(G, H_{k,\ell,\dots}) = p(G; k, \ell, \dots)$$

for polynomial $p(G)$?

Characterizing simple graph sequences $(H_{k,\ell,\dots})$ with this property gives straightforward characterization for multigraph sequences too (allowing multiple edges & loops).

Example 1

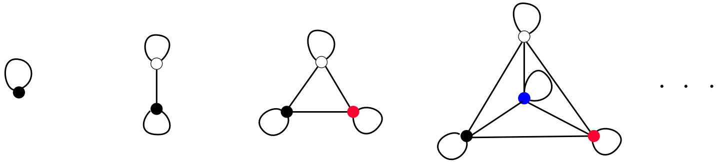


$$(K_k)$$

$$\hom(G, K_k) = P(G; k)$$

chromatic polynomial

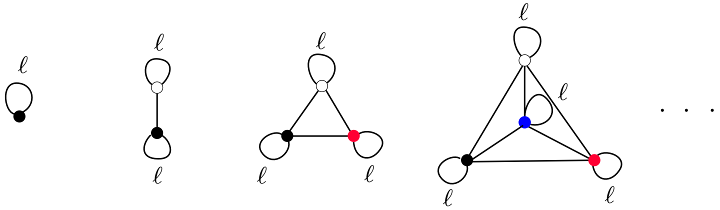
Example 2



$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

Example 3



$$(K_k^\ell)$$

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= k^{c(G)}(\ell-1)^{r(G)}T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$ (Tutte polynomial)

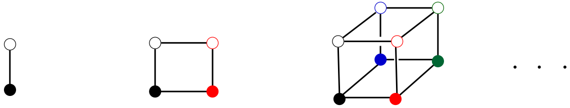
Example 4

$$(K_1^1 + K_{1,k})$$

$$\hom(G, K_1^1 + K_{1,k}) = I(G; k)$$

independence polynomial

Example 5



$$(K_2^{\square k}) = (Q_k) \text{ (hypercubes)}$$

Proposition (Garijo, G., Nešetřil, 2013+)

$\hom(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

Since $\hom(G_1 \cup G_2, H) = \hom(G_1, H)\hom(G_2, H)$, suffices to consider *connected* G.

Example

- (K_k), (K_k^1). $(\overline{kK_2})$ are strongly polynomial
- (K_k^ℓ) is strongly polynomial (in k, ℓ)
- (Q_k) not strongly polynomial (but polynomial in k and 2^k)
- (C_k), (P_k) not strongly polynomial (but eventually polynomial in k)

Subgraph criterion for strongly polynomial

$$H_k \text{ simple:} \qquad \hom(G, H_k) = \sum_{\substack{S \subseteq_{\mathrm{i}} H_k \\ |V(S)| \leq |V(G)|}} \mathrm{sur}_{\mathsf{V}}(G, S)$$

$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V}}(G, S)\ \#\{\text{induced copies of } S \text{ in } H_k\}$$

Proposition (de la Harpe & Jaeger 1995)

(H_k) *is strongly polynomial* $\iff$
$\forall$*connected* S $\#\{\text{induced subgraphs} \cong S \text{ in } H_k\}$ *polynomial in* k

Subgraph criterion for strongly polynomial

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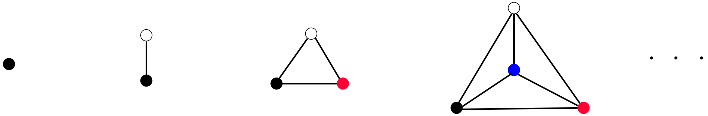
$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V}}(G, S) \, \#\{\text{induced copies of } S \text{ in } H_k\}$$

(for each S want this polynomial in k)

Proposition (de la Harpe & Jaeger 1995)

(H_k) is strongly polynomial $\iff$
$\forall$connected S $\#\{\text{induced subgraphs} \cong S \text{ in } H_k\}$ *polynomial in k*

Example: chromatic polynomial

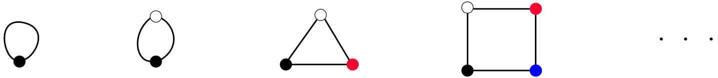


$$\hom(G, K_k) = P(G; k) = \sum_{1 \le j \le \min\{|V(G)|, k\}} \mathrm{sur}_V(G, K_j)\binom{k}{j}$$

$$= \sum_{1 \le j \le |V(G)|} \mathrm{sur}_V(G, K_j)\binom{k}{j},$$

as $\binom{k}{j} = 0$ when $j > k \ge |V(G)|$.

Eventually polynomial but not strongly polynomial



(C_k)

$$\mathrm{hom}(G, C_k) = \sum_{1 \le j \le \min\{|V(G)|, k-1\}} \mathrm{sur}_V(G, P_j)\, k + \mathrm{sur}_V(G, C_k)$$

$\mathrm{hom}(C_3, C_3) = 6$, $\mathrm{hom}(C_3, C_k) = 0$ when $k = 2$ or $k \geq 4$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

If (H_k) *is strongly polynomial and* H_k *simple, then*

- $(\overline{H_k})$ *(complements),*
- $(L(H_k))$ *(line graphs),*

are also strongly polynomial.
Also, (ℓH_k) *is strongly polynomial (in* k *and* ℓ*).*

Proposition (Garijo, G., Nešetřil, 2013+)

If (H_k) *is strongly polynomial, at most one loop each vertex of* H_k*, then*

- (H_k^0) *(remove all loops)*
- (H_k^1) *(add loops to make 1 loop each vertex)*

are also strongly polynomial.
More generally, (H_k^ℓ) *is strongly polynomial (in* k *and* ℓ*).*

Proposition

If (F_j), (H_k) are strongly polynomial, then

- $(F_j \cup H_k)$ *(disjoint union)*
- $(F_j + H_k)$ *(join)*
- $(F_j \times H_k)$ *(direct/categorical product)*
- $(F_j[H_k])$ *(lexicographic product)*

are strongly polynomial (in j and k).

Example

Beginning with trivially strongly polynomial sequence (K_1), following are also strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$
- complement: (K_k) *(chromatic polynomial)*
- loop-addition: (K_k^ℓ) *(Tutte polynomial)*
- join: $(K_{k-j}^1 + K_j^\ell)$ *(Averbouch–Godlin–Makowsky polynomial – includes Tutte polynomial, satisfies three-term recurrence in* $\backslash uv, /uv$ *and* $-u-v$*)*

Question

Strongly polynomial sequences:

- $(\overline{K_j} + \overline{K_k}) = (K_{j,k})$
- $(L(K_{j,k})) = (K_j \Box K_k)$ *(Rook's graph)*

(F_j), (H_k) strongly polynomial $\Rightarrow$ $(F_j \Box H_k)$ strongly polynomial?

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \dots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

- Johnson graphs $D = \{k - 1\}$
- Kneser graphs $D = \{0\}$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

For every ℓ, D, sequence $(J_{k,\ell,D})$ is strongly polynomial (in k).

Question

Can generalized Johnson graphs be generated from simpler sequences by any of the constructions described in de la Harpe & Jaeger (1995) and Garijo, Goodall & Nešetřil (2013+)?

Simple graph sequence (H_k) strongly polynomial iff

- $\forall G\ \exists$ polynomial $p(G)\ \forall k \in \mathbb{N} \quad \hom(G, H_k) = p(G; k)$
- $\forall F\ \exists$ polynomial $q(F)\ \forall k \in \mathbb{N} \quad \mathrm{ind}(F, H_k) = q(F; k)$

Unary operations $\widetilde{\ }$ and binary operations $*$ such that if simple graph sequences (F_j) and (H_k) are strongly polynomial then

- $(\widetilde{H}_k)$ is strongly polynomial (e.g. complement, line graph)
- $(F_j * H_k)$ is strongly polynomial in j, k (e.g. join, lexicographic product)

Satisfaction sets

Quantifier-free formula ϕ with n free variables ($\phi \in \mathrm{QF}_n$) with symbols from relational structure $\mathbf{H}$ with domain $V(\mathbf{H})$.

Satisfaction set $\phi(\mathbf{H}) = \{(v_1, \ldots, v_n) \in V(\mathbf{H})^n : \mathbf{H} \models \phi\}$.

e.g. for graph structure H (symmetric binary relation $x \sim y$ interpreted as x adjacent to y), and given graph G on n vertices,

$$\phi = \phi_G = \bigwedge_{ij \in E(G)} (v_i \sim v_j)$$

$$\phi_G(H) = \{(v_1, \ldots, v_n) : \ i \mapsto v_i \text{ is a homomorphism } G \to H\}$$

$$|\phi_G(H)| = \mathrm{hom}(G, H).$$

Strongly polynomial sequences of structures

Definition

Sequence $(\mathbf{H}_k)$ of relational structures strongly polynomial iff
$\forall \phi \in QF \; \exists$ polynomial $r(\phi) \; \forall k \in \mathbb{N} \quad |\phi(\mathbf{H}_k)| = r(\phi; k)$

Lemma

Equivalently,

- $\forall \mathbf{G} \; \exists$ *polynomial* $p(\mathbf{G}) \; \forall k \in \mathbb{N} \quad \hom(\mathbf{G}, \mathbf{H}_k) = p(\mathbf{G}; k)$, *or*
- $\forall \mathbf{F} \; \exists$ *polynomial* $q(\mathbf{F}) \; \forall k \in \mathbb{N} \quad \mathrm{ind}(\mathbf{F}, \mathbf{H}_k) = q(\mathbf{F}; k)$.

Transitive tournaments $(\vec{T}_k)$ strongly polynomial sequence of digraphs (e.g. count induced substructures).

Graphical QF interpretation schemes

$$I: \text{ Relational } \sigma\text{-structures } \mathbf{A} \quad \longrightarrow \quad \text{Graphs } H$$

Lemma

There is

$$\widetilde{I}: \phi \in \mathrm{QF}(\mathrm{Graphs}) \quad \longmapsto \quad \widetilde{I}(\phi) \in \mathrm{QF}(\sigma\text{-}\mathit{structures})$$

such that

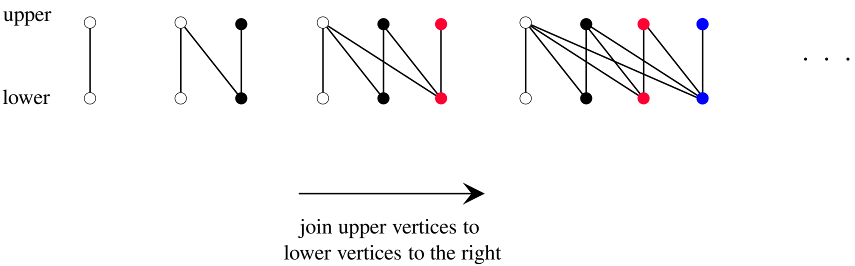
$$\phi(I(\mathbf{A})) = \widetilde{I}(\phi)(\mathbf{A})$$

In particular, $(\mathbf{A}_k)$ *strongly polynomial* $\quad \Rightarrow \quad (H_k) = (I(\mathbf{A}_k))$ *strongly polynomial.*

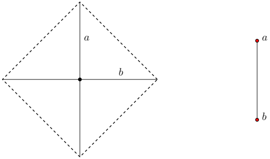
From graphs to graphs

- All previous constructions (complementation, line graph, disjoint union, join, direct product,...) special cases of interpretation schemes I from Marked Graphs (added unary relations) to Graphs.
- Cartesian product and other more complicated graph products are special kinds of such interpretation schemes too.

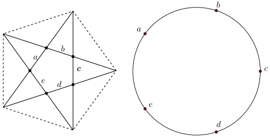
- Generalized Johnson graphs ($J_{k,\ell,D}$) arise as QF interpretations of transitive tournaments $\vec{T}_k$
- Half-graphs are QF interpretations of a transitive tournament together with "marks" (unary relations used to specfiy "upper" + "lower" vertices) and so form a strongly polynomial sequence.



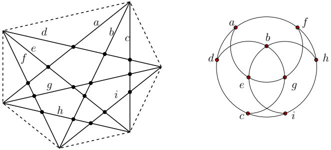
- Intersection graphs of chords of a k-gon form a strongly polynomial sequence



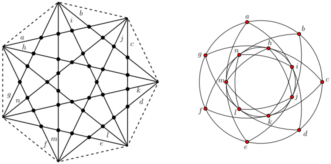
(a) Square



(b) Pentagon



(c) Hexagon



(d) Heptagon

Conjecture

All strongly polynomial sequences of graphs (H_k) can be obtained by QF interpretation of a "basic sequence" (disjoint union of marked transitive tournaments of size polynomial in k).

Prime power $q = p^d \equiv 1 \pmod 4$
Paley graph P_q =Cayley($\mathbb{F}_q$, non-zero squares),
Quasi-random graphs: $\hom(G, P_q)/\hom(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Proposition (Corollary to result of de la Harpe & Jaeger, 1995)

$\hom(G, P_q)$ *is polynomial in* q *for series-parallel* G.
e.g. $\hom(K_3, P_q) = \frac{q(q-1)(q-5)}{8}$

Prime $q \equiv 1 \pmod 4$, $q = 4x^2 + y^2$, [Evans, Pulham, Sheehan, 1981]:

$$\hom(K_4, P_q) = \frac{q(q-1)}{1536}\left((q-9)^2 - 4x^2\right)$$

Is $\hom(G, P_q)$ polynomial in q and x for all graphs G?

Theorem (G., Nešetřil, Ossona de Mendez , 2014+)

If (H_k) *is strongly polynomial then there are only finitely many terms belonging to a quasi-random sequence of graphs.*

- When is $(\mathrm{Cayley}(A_k, B_k))$ polynomial in $|A_k|, |B_k|$, where $B_k = -B_k \subseteq A_k$?
  e.g. For $D \subset \mathbb{N}$, sequence $(\,\mathrm{Cayley}(\mathbb{Z}_k, \pm D)\,)$ is polynomial iff D is finite or cofinite. (de la Harpe & Jaeger, 1995)
- Can (H_k) be verified to be strongly polynomial by testing $\hom(G, H_k)$ for G only in a restricted class of graphs? (yes, for connected graphs – but for a smaller class?)
- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial and independence polynomial?
- Develop similar theory for $\hom(H_k, G)$ (e.g. $\hom(C_k, G) = \sum \lambda^k$, λ eigenvalues of G, determines characteristic polynomial of G by its roots).

감사합니다

!!!

Three papers

- P. de la Harpe and F. Jaeger, Chromatic invariants for finite graphs: theme and polynomial variations, *Lin. Algebra Appl.* **226–228** (1995), 687–722

  Defining graphs invariants from counting graph homomorphisms. Examples. Basic constructions.

- D. Garijo, A. Goodall, J. Nešetřil, Polynomial graph invariants from homomorphism numbers. 40pp. arXiv: 1308.3999 [math.CO]
  Further examples. New construction using tree representations of graphs.

- A. Goodall, J. Nešetřil, P. Ossona de Mendez, Strongly polynomial sequences as interpretation of trivial structures. 17pp. Preprint.
  General relational structures: counting satisfying assignments for quantifier-free formulas. Building new polynomial invariants by interpretation of "trivial" sequences of marked tournaments.