Graph invariants, homomorphisms, and the Tutte polynomial

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2 Flows and tensions

2.1 Orientations

An undirected graph $G = (V, E)$ can be made into a digraph in $2^{|E|}$ ways: for each edge $uv \in E$ we decide to direct u towards v, or to direct v towards u. If the edge is a loop, i.e. $u = v$, then we still think of there being two opposite ways to orient the loop – this is a matter of convenience for later definitions (and makes sense when we talk about orienting plane graphs, where the two possible directions can indeed be distinguished).

We orient a graph in order to extract structural properties of the underlying undirected graph, but the orientation that is chosen is arbitrary: the results obtained are independent of this choice. (The reader may recall the rôle played by an orientation of G in proving Kirchhoff's Matrix Tree Theorem, which gives an expression for the number of spanning trees of G.)

Suppose then we are given an orientation ω of $G = (V, E)$. By this we mean that ω assigns a direction to each edge $uv \in E$, either $u \xrightarrow{\omega} v$ or $u \xleftarrow{\omega} v$. We write G^ω for the digraph so obtained. For $U \subset V$ let $\omega^+(U)$ denote the set of the edges which begin in U and end outside U in the digraph G^ω, i.e., $\omega^+(U) = \{uv \in E : u \in U, v \in V \setminus U \; u \xrightarrow{\omega} v\}$. The set $\omega^-(U) = \omega^+(V \setminus U)$ comprises edges which in G^ω begin outside U and end in U.

For a vertex $v \in V$ the set $\omega^+(\{v\})$ consists of those edges directed out of v by the orientation ω and $\omega^-(\{v\})$ is the set of edges directed into v. The *indegree* of a vertex v in G^ω is $|\omega^-(\{v\})|$ and its *outdegree* is $|\omega^+(\{v\})|$.

If G is a plane graph then each orientation ω of G determines an orientation ω^* of its dual G^*. This orientation is obtained by giving an edge e^* of G^* the orientation that is obtained from that of e by rotating it 90^o clockwise: the edge e^* travels from the face to the left of e to the face to the right of e.

More formally, given an orientation ω of the plane graph $G = (V, E)$ we define the orientation ω^* of $G^* = (V^*, E^*)$ as follows. Let V^* be the set of faces of the embedded graph G. For each arc $u \xrightarrow{\omega} v$ of G^ω, suppose uv lies on the boundary of faces X and Y. Suppose further that X is the face that would be traversed anticlockwise if the direction of $u \xrightarrow{\omega} v$ were followed all the way round it (so that Y would be traversed in a clockwise direction following the direction given by $u \xrightarrow{\omega} v$). Then under the orientation ω^* we direct edge XY of G^* as the arc $X \xrightarrow{\omega^*} Y$.

See Fig.1

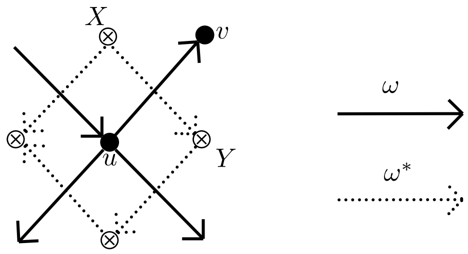

Figure 1: Dual orientation ω^* of an orientation ω of a plane graph

Question 1

(i) What is the dual orientation of ω^*?

(ii) If C is a circuit of G all of whose edges follow the same direction under orientation ω (i.e. it is cyclically oriented) then what is the dual of C and how is it oriented under orientation ω^*?

2.2 Circuits and cocircuits

We use terminology for graphs here that corresponds to viewing a graph $G = (V, E)$ in terms of its cycle matroid; the sense of "circuit" and "cycle" therefore differs from traditional graph theoretical usage.

A *cycle* is a set of edges defining a spanning subgraph of G all of whose vertex degrees are even (i.e., an Eulerian subgraph). A *circuit* is an inclusion-minimal cycle (i.e., a connected 2-regular subgraph). A cycle is the edge-disjoint union of circuits. A subset of edges is *dependent* in a graph $G = (V, E)$ if it contains a cycle and *independent* otherwise. An independent set of edges forms a forest. A maximal independent set of edges (add an edge and a cycle is formed) of a connected graph is a spanning tree.

A *cutset* K is a subset of edges defined by a bipartition of V, i.e., $K = \{uv \in E : u \in U, v \in V \setminus U\}$ where $U \subseteq V$. A *cocircuit* (or *bond*) is an inclusion-minimal cutset of $G = (V, E)$. A cocircuit of a connected graph is a cutset $\{uv \in E : u \in U, v \in V \setminus U\}$ with the additional property that the induced subgraphs $G[U]$ and $G[V \setminus U]$ are both connected. The rank of a graph decreases when removing a cutset. A cutset K is a cocircuit if and only if deleting K produces exactly one extra connected component, i.e., in this case $r(G\backslash K) = r(G) - 1$.

The nullity of the graph G is defined by $n(G) = |E| - r(G)|$. The nullity of G decreases when contracting the edges of a cycle of G into a single vertex, and for a circuit C we have $n(G/C) = n(G) - 1$. In terms of the cycle matroid of G, a circuit C is a minimal set of dependent edges: removing an edge from C destroys the cycle that makes C dependent.

Question 2
A bridge in a graph G forms a cutset of G by itself. Dually, a loop in G forms a cycle of G by itself. Show that

(i) an edge e is a bridge in G if and only if e does not belong to any circuit of G.

(ii) an edge e is a loop in G if and only if e does not belong to any cocircuit of G.

A subset B is a cocircuit of a connected graph G if and only if contracting all edges not in B (and deleting any isolated vertices that result) produces a "bond-graph", consisting of two vertices joined by $|B|$ parallel edges. (A subset K is a cutset of G if and only if the result is a graph whose blocks are bond-graphs – the vertices in this graph correspond to the connected components of $G \setminus K$.) Likewise, a subset C is a circuit of G if and only if deleting all the edges not in C (and deleting any isolated vertices that result) produces a cycle-graph (2-regular) on $|C|$ edges.

Question 3
Show that if $G = (V, E)$ is a plane graph and G^* is its dual then a subset of edges B is a cocircuit of G if and only if B is a circuit of G^*. (Assume the Jordan Curve Theorem: a simple closed curve – such as that bounding a circuit in a plane graph – partitions the plane minus the curve into an interior region bounded by the curve and an exterior region.)

A *spanning tree* of a connected graph G is a maximal set of independent edges: adding an edge creates a cycle. A spanning tree of G is a basis of the cycle matroid of G. More generally, when G is not connected, a maximal set of independent edges is a maximal spanning forest of G (add an edge and it is no longer a forest).

Suppose $G = (V, E)$ is connected and T is a spanning tree of G. Then

(i) for each $e \in E \setminus T$ there is a unique circuit of G contained in $T \cup \{e\}$, which we shall denote by $C_{T,e}$, and

(ii) for each $e \in T$ there is a unique cocircuit contained in $E \setminus T \cup \{e\}$, which we shall denote by $B_{T,e}$.

Let C be a circuit of G. The two possible cyclic orderings of the edges of C define two cyclic orientations of the edges of C. Choose one of these orientations arbitrarily, making a directed cycle $\overrightarrow{C}$. Define C^+ to be the set of edges whose orientation in G^ω is the same as that in $\overrightarrow{C}$, and define C^- to be the set of those edges directed in G^ω in the opposite direction to that in $\overrightarrow{C}$.

This signing extends to cycles (Eulerian subgraphs) more generally, since any cycle is a disjoint union of circuits: when the cycle is the union of k edge-disjoint circuits there are 2^k choices for signing it.

Similarly, for a cocircuit (bond) B of G, defined by $U \subset V$ such that $B = \{uv \in E : u \in U, v \notin U\}$, we orient the bond B by directing edges from U to $V \setminus U$ to make $\overrightarrow{B}$. (Again there are two choices of orientation, depending on which side of the cut we nominate to be U and which side $V \setminus U$.) We then define B^+ and B^- in an analogous way to circuits. Clearly this procedure of signing cocircuits extends to cutsets more generally by directing edges from one side of the cut to the other. (Alternatively, a cutset is a disjoint union of cocircuits (why?), so in a similar way to cycles we can sign a cutset by signing its constituent cocircuits.)

We have already encountered signed cutsets in Section 2.1: for a subset $U \subset V$ the set $\omega^+(U)$ of edges that begin in U and terminate outside U comprise the positive elements of the cocircuit defined by U, and $\omega^-(U) = \omega^+(V \setminus U)$ the negative elements.

In this way, for a given orientation of G as a digraph G^ω, we have separated the edge sets of (co)circuits into positive and negative elements. In fact, given G and its set of (co)circuits, if the partition of each (co)circuit into positive and negative elements is given, then we can recover the orientation of edges (provided the way the (co)circuits have been signed is consistent with some orientation – what conditions are required for this to be the case?).

Question 4
A matroid is *regular* if there is an orientation of its circuits and cocircuits such that for all circuits C and all cocircuits B

$$|C^+ \cap B^+| + |C^- \cap B^-| = k \Leftrightarrow |C^+ \cap B^-| + |C^- \cap B^+| = k.$$

Explain why this statement holds for graphic matroids.

Definition 1. *Let C be a signed circuit of an oriented graph G^ω on edge set E. The* signed characteristic vector $\overrightarrow{\chi}_C \in \{0, \pm 1\}^E$ *of C is defined by*

$$\overrightarrow{\chi}_C(e) = \begin{cases} 1 & \text{if } e \in C^+, \\ -1 & \text{if } e \in C^- \\ 0 & \text{if } e \notin C. \end{cases}$$

The signed characteristic vector $\overrightarrow{\chi}_B$ of a signed cocircuit B is similarly defined.

A fundamental relationship between signed characteristic vectors of circuits and cocircuits (and of cycles and cutsets more generally) is given by the following:

Proposition 2. *The signed characteristic vector of a circuit C is orthogonal to the signed characteristic vector of a cocircuit B:*

$$\sum_{e \in E} \overrightarrow{\chi}_B(e)\overrightarrow{\chi}_C(e) = 0.$$

Proof. Given a cocircuit K with positive elements $\omega^+(U)$ and negative elements $\omega^-(U)$, the inner product $\sum_{e\in E} \overrightarrow{\chi}_K(e)\overrightarrow{\chi}_C(e)$ is the number of edges of the circuit C going from U to $V \setminus U$ in its circuit-orientation, minus the number of edges going from $V \setminus U$ to U, and this is equal to zero. (In the simple closed walk that follows the edges of the circuit, for each edge followed in the direction from U to $V \setminus U$ there is a corresponding edge followed in the reverse direction from $V \setminus U$ to U.) □

2.3 The incidence matrix of an oriented graph

We suppose still that we are given an orientation ω of the graph $G = (V, E)$.

Definition 3. *The* incidence matrix *of an oriented graph G^ω is the matrix $D = (d_{v,e}) \in \{0, \pm 1\}^{V \times E}$ whose (v, e)-entry is defined by*

$$d_{v,e} = \begin{cases} +1 & \text{if } e \text{ is directed out of } v \text{ by } \omega, \\ -1 & \text{if } e \text{ is directed into } v \text{ by } \omega, \\ 0 & \text{if } e \text{ is not incident with } v, \text{ or } e \text{ is a loop on } v. \end{cases}$$

A loop e corresponds to a zero column of D indexed by e (the fact that under any orientation the loop e is both going out of and going into v implies any flow along this edge is self-cancelling); each column of D indexed by an ordinary edge or bridge contains one entry $+1$, one entry -1, and remaining entries all 0.

The row of D indexed by u is equal to $\overrightarrow{\chi}_{\omega^+(\{u\})\cup\omega^-(\{u\})}$ (regarded as a row vector). If G is connected then if we delete any row of D the remaining rows form a basis for the signed characteristic vectors of cutsets. This is because

$$\overrightarrow{\chi}_{\omega^+(U)\cup\omega^-(U)} = \sum_{u\in U} \overrightarrow{\chi}_{\omega^+(\{u\})\cup\omega^-(\{u\})},$$

and we may choose U to not contain the vertex whose row has been deleted. More generally, for any graph G there are $r(G)$ rows of D spanning signed characteristic vectors of cutsets, which can be obtained by deleting, for each component of G, one row indexed by a vertex in the component.

Let A be an additive Abelian group (for us A will either be $\mathbb{Z}$ or finite). Scalar multiples of a $\{0, \pm 1\}$-vector by an element of A are defined by using the identities $0a = 0, 1a = a$ and $(-1)a = -a$ for each $a \in A$. The Abelian group A is a $\mathbb{Z}$-module, with the action of $\mathbb{Z}$ defined inductively by $ta = (t-1)a + a$ for integer $t > 0$ and $ta = -|t|a$ (inverse of $|t|a$ in A) for integer $t < 0$.

The set of vectors with entries in A indexed by E is denoted by A^E, and likewise A^V those vectors with entries indexed by V. We shall think of elements of A^E interchangeably as elements of the additive group formed by taking the $|E|$-fold direct sum of A with itself, as vectors indexed by E, or as functions $\phi : E \to A$.

The incidence matrix defines a homomorphism $D : A^E \to A^V$ between additive groups, and its transpose likewise a homomorphism $D^T : A^V \to A^E$. For each $\phi : E \to A$,

$$(D\phi)(v) = \sum_{\substack{e=uv \\ u \xleftarrow{\omega} v}} \phi(e) - \sum_{\substack{e=uv \\ u \xrightarrow{\omega} v}} \phi(e).$$

The map $D : A^E \to A^V$ is called the *boundary*, assigning the net flow to each vertex from the given mapping $\phi : E \to A$.

For $\kappa : V \to A$ and edge $e = uv$,

$$(D^T\kappa)(e) = \begin{cases} \kappa(v) - \kappa(u) & \text{if } u \xleftarrow{\omega} v \\ \kappa(u) - \kappa(v) & \text{if } u \xrightarrow{\omega} v. \end{cases}$$

By the first isomorphism theorem for groups we have $\operatorname{im} D \cong A^E / \ker D$ and $\operatorname{im} D^T \cong A^V / \ker D^T$.

Proposition 4. *Let G be a graph with connected components on vertex sets $V_1, \ldots, V_{c(G)}$.*

(i) The incidence mapping $D : A^E \to A^V$ has image

$$\operatorname{im} D = \{\kappa : V \to A;\ \sum_{v \in V_i} \kappa(v) = 0, \text{ for each } 1 \leq i \leq c(G)\} \cong A^{r(G)}.$$

(ii) The transpose $D^T : A^V \to A^E$ has kernel

$$\ker D^T = \{\kappa : V \to A;\ \kappa \text{ constant on } V_i, \text{ for each } 1 \leq i \leq c(G)\} \cong A^{c(G)}.$$

Proof. (i) Given $\phi : E \to A$ we have

$$\begin{aligned} \sum_{v \in V_i} (D\phi)(v) &= \sum_{v \in V_i} \sum_{e \in E} d_{v,e}\phi(e) \\ &= \sum_{e \in E} \phi(e) \sum_{v \in V_i} d_{v,e} \\ &= 0, \end{aligned}$$

the last line since the entries $\{d_{v,e} : v \in V_i\}$ in the column c_e of D are either all zero (when e is not an edge in the component on V_i), or contain precisely $+1$ and -1 as non-zero elements.

Conversely, suppose that $\kappa : V \to A$ is such that $\sum_{v \in V_i} \kappa(v) = 0$ for each $1 \leq i \leq c(G)$. For given i, choose any $u \in V_i$. Then, letting κ_i denote the restriction of κ to V_i,

$$\kappa_i = \sum_{v \in V_i} \kappa(v)\delta_v = \sum_{v \in V_i \setminus \{u\}} \kappa(v)(\delta_v - \delta_u)$$

where δ_v is defined by $\delta_v(w) = 1$ is $w = v$ and $\delta_v(w) = 0$ otherwise. Since $G[V_i]$ is connected, for each $v \in V_i$ there is a path from u to v, say $u = v_0, e_1, v_1, \ldots, v_{\ell-1}, e_\ell, v_\ell = v$ and

$$\delta_v - \delta_u = (\delta_{v_\ell} - \delta_{v_{\ell-1}}) + \cdots + (\delta_{v_1} - \delta_{v_0}) = D(\pm\delta_{e_\ell}) + \cdots + D(\pm\delta_{e_1}),$$

where $\delta_e(f) = 1$ if $e = f$ and 0 otherwise, and the signs are chosen according to whether the directed path from u to v follows the orientation ω or goes against it. Hence $\delta_v - \delta_u \in \operatorname{im} D$ for each $v \in V_i$, whence $\kappa_i \in \operatorname{im} D$ also. This implies finally that κ itself belongs to $\operatorname{im} D$.

(ii) Suppose that $\kappa : V \to A$ is such that $D^T\kappa = 0$. For an edge $e = uv$ with orientation $u \stackrel{\omega}{\longleftarrow} v$, $(D^T\kappa)(e) = \kappa(v) - \kappa(u) = 0$, so that κ takes the same value on the endpoints of any edge. If u and w are in the same component of G then there is a walk starting at u and finishing at w and so $\kappa(u) = \kappa(w)$.

Conversely, if κ is constant on every component then $D^T\kappa = 0$.

□

The subgroups $\ker D^T$ and $\operatorname{im} D$ of A^V are of less interest from the point of view of their relationship to the combinatorial properties of the graph G than the subgroups $\ker D$ and $\operatorname{im} D^T$ of A^E. From Proposition 4 we know that as additive groups $\ker D \cong A^{n(G)}$ and $\operatorname{im} D^T \cong A^{r(G)}$. The combinatorial interest comes from the fact that there are generating sets for these groups associated with circuits and cocircuits of G, and that further structural properties of $\ker D$ and $\operatorname{im} D^T$ (namely properties of the intersections $\ker D \cap B^E$ and $\operatorname{im} D \cap B^E$ where $B \subset A$) correspond to combinatorial features of the graph. We shall be concerned in particular with the case $B = A \setminus \{0\}$, and when $A = \mathbb{R}$ also with the case $B = \mathbb{Z}$.

2.4 A-flows and A-tensions

From now on we assume that A is a commutative ring and we consider A^E and its subgroups $\ker D$ and $\operatorname{im} D^T$ as modules over A.

When $A = \mathbb{Z}$ we take ordinary integer multiplication. When A is finite, by the classification theorem for finite Abelian groups A takes the form $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_r}$, where $2 \leq k_1 \mid k_2 \mid \cdots \mid k_r$ (the notation $a \mid b$ meaning that a divides b), where k_r is the least common multiple of the orders of the elements of A. Componentwise multiplication then endows A with the structure of a commutative ring $R \cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_r}$. Note however that if A is the r-fold direct sum of $\mathbb{Z}_p$ for prime p then there is another natural choice of multiplication, namely that which makes A the finite field $\mathbb{F}_{p^r}$.

Let us start by defining flows[1] on a graph in what is the usual way, by

[1] In other sources what we call a "nowhere-zero flow" is often just called a "flow", while what we have chosen to call a "flow" is called a "circulation". Compare too a "proper colouring" of vertices if a graph, which is conventionally just called a "colouring", while an arbitrary assignment of colours to vertices is given some other name or "colouring" is qualified by a parenthetical "not necessarily proper".

stipulating that the Kirchhoff condition holds at each vertex. We shall then derive various other equivalent definitions.

Definition 5. *An* A-flow *of* G *is a mapping* $\phi : E \to A$ *such that*

$$\sum_{e \in \omega^+(\{v\})} \phi(e) - \sum_{e \in \omega^-(\{v\})} \phi(e) = 0 \qquad \text{for each } v \in V.$$

A nowhere-zero *A-flow is an A-flow* $\phi : E \to A$ *with the additional property that* $\phi(e) \neq 0$ *for every* $e \in E$.

In other words, an A-flow ϕ as a vector is an element of $\ker D$, since the signed characteristic vectors $\overrightarrow{\chi}_{\omega^+(\{v\}) \cup \omega^-(\{v\})}$ are the rows of D.

For any $U \subseteq V$ we have

$$\sum_{u \in U} \overrightarrow{\chi}_{\omega^+(\{u\}) \cup \omega^-(\{u\})} = \overrightarrow{\chi}_{\omega^+(U) \cup \omega^-(U)},$$

since when $e = uv \in E$ has both $u \in U$ and $v \in U$ we have $e \in \omega^+(\{u\})$ and $e \in \omega^-(\{v\})$, or vice versa, so that $\overrightarrow{\chi}_{\omega^+(\{u\}) \cup \omega^-(\{u\})}(e) + \overrightarrow{\chi}_{\omega^+(\{v\}) \cup \omega^-(\{v\})}(e) = 0$.

For any $U \subseteq V$ we have

$$\sum_{u \in U} \left(\sum_{e \in \omega^+(\{u\})} \phi(e) - \sum_{e \in \omega^-(\{u\})} \phi(e) \right)$$
$$= \sum_{e \in \omega^+(U)} \phi(e) - \sum_{e \in \omega^-(U)} \phi(e)$$

since when $e = uv \in E$ has both $u \in U$ and $v \in U$ we have $e \in \omega^+(\{u\})$ and $e \in \omega^-(\{v\})$ so that cancellation of $\phi(e)$ with $-\phi(e)$ occurs for the edge e.

Hence it is equivalent to define an A-flow as a mapping $\phi : E \to A$ such that

$$\sum_{e \in B^+} \phi(e) - \sum_{e \in B^-} \phi(e) = 0 \qquad \text{for every cocircuit } B \text{ of } G.$$

Introduce a bilinear form $\langle\, , \,\rangle$ on A^E by setting

$$\langle \phi, \psi \rangle = \sum_{e \in E} \phi(e)\psi(e).$$

In this notation, Proposition 2 says that $\langle \overrightarrow{\chi}_B, \overrightarrow{\chi}_C \rangle = 0$ for each signed bond B and signed circuit C.

Since the signed characteristic vectors $\overrightarrow{\chi}_{B_v}$ of the signed bonds $B_v = \omega^+(v) \cup \omega^-(v)$ span the characteristic vectors $\overrightarrow{\chi}_B$ of all bonds B, it is equivalent to define ϕ to be an A-flow if and only if

$$\langle \phi, \overrightarrow{\chi}_B \rangle = 0 \qquad \text{for each signed bond } B.$$

Since signed characteristic vectors of bonds are orthogonal to signed characteristic vectors of circuits, ϕ is an A-flow of G if

$$\phi = \sum_{C\in\mathcal{C}} a_C \overrightarrow{\chi}_C \quad \text{for some } a_C \in A \text{ indexed by } C \in \mathcal{C}.$$

The converse is immediate when A is finite or a field: in the first case by counting (we know that $\ker D \cong A^{n(G)}$ and there are $n(G)$ linearly independent signed characteristic vectors $\overrightarrow{\chi}_C$) and in the second case by orthogonal decomposition of vector spaces. When $A = \mathbb{Z}$ the fact that all flows take the form $\phi = \sum_{C\in\mathcal{C}} a_C \overrightarrow{\chi}_C$ for $a_C \in \mathbb{Z}$ amounts to the fact that the signed characteristic vectors $\overrightarrow{\chi}_C$ form an integral basis for $\ker D$ as a lattice in $\mathbb{R}^E$.

Let $\mathcal{B}$ and $\mathcal{C}$ denote respectively the set of signed bonds and signed circuits of the oriented graph G^ω on edge set E. The set of A-*flows* of G is given by

$$\mathcal{Z}_A = \{\sum_{C\in\mathcal{C}} a_C \overrightarrow{\chi}_C \,|\, a_C \in A\}.$$

A graph G has a *nowhere-zero k-flow* if there is a flow $\phi \in \mathcal{Z}_\mathbb{Z}$ such that $0 < |\phi(e)| < k$ for all $e \in E$, and a *nowhere-zero A-flow* if there is a flow $\phi \in \mathcal{Z}_A$ such that $\phi(e) \neq 0$ for all $e \in E$.

The row space of the incidence matrix of G^ω is spanned by the signed characteristic vectors of cocircuits of G. The dual notion to flows is that of tensions (also known as *coflows*), which are defined as elements of $\operatorname{im} D^T$:

Definition 6. *Let $\mathcal{B}$ denote the set of signed bonds of an oriented graph G^ω on edge set E.*

*The set of A-*tensions *is defined by*

$$\mathcal{K}_A = \{\sum_{B\in\mathcal{B}} a_B \overrightarrow{\chi}_B \,|\, a_B \in A\}.$$

By Proposition 2 the signed characteristic vectors of circuits and cocircuits are orthogonal (as vectors over $\mathbb{Z}$). Once multiplication is defined making A into a ring, this extends to the following key relationship between flows and tensions:

Theorem 7. *Suppose A is a commutative ring. If ϕ is an A-flow of a graph G and θ is an A-tension of G then ϕ and θ are orthogonal as vectors over A:*

$$\sum_{e\in E} \phi(e)\theta(e) = 0.$$

An A-flow or A-tension whose value on each edge of G belongs to $B \subseteq A$ is called a B-flow or B-tension respectively. In the next section we shall be particularly interested in the case $B = A \setminus 0$.

Proposition 8. *A vector is a $\mathbb{Z}_2$-flow if and only if it is the characteristic function of an Eulerian subgraph of G and is a $\mathbb{Z}_2$-tension if and only if its is the characteristic vector of a cutset of G.*

When working over $\mathbb{Z}_2$ the signed characteristic functions of signed (co)circuits become characteristic functions of (co)circuits. The set of $\mathbb{Z}_2$-flows is a binary vector space called the *cycle space* of G, comprising characteristic vectors of Eulerian subgraphs of G, and the set of $\mathbb{Z}_2$-tensions is called the *cocycle space* of G, comprising characteristic vectors of cutsets of G.

Question 5

(i) To what does an integer 2-flow of G correspond? When does G has a nowhere-zero 2-flow?

(ii) Dually, when does G have a nowhere-zero 2-tension?

(iii) Is it true that G has a nowhere-zero 2-flow if and only if G has a nowhere-zero $\mathbb{Z}_2$-flow? And dually, what is the analogous statement for nowhere-zero 2-tensions and nowhere-zero $\mathbb{Z}_2$-tensions?

3 Tensions and colourings

An *A-potential* of G is a mapping $\kappa : V \to A$ and can be thought of as a (not necessarily proper) vertex colouring of G with colours the elements of A. Given an orientation ω of G, the mapping $D^T\kappa$ is called the *potential difference* or *coboundary* of κ.

We identify colourings of the vertices of G, where the colours are taken in A, with the corresponding A-potential of G. An A-tension of G corresponds to $|A|^{c(G)}$ different A-colourings of G: to each $\theta \in \mathcal{K}_A$ corresponds $|A|^{c(G)}$ colourings $\kappa : V \to A$ with $D^T\kappa = \theta$. This relationship of tensions to vertex colourings is what underlies the duality between colourings and flows, as we shall see.

For a proper vertex A-colouring the corresponding A-tension is nowhere-zero. This is a basic observation linking flows and colourings and leads to the following:

Proposition 9. *Let G be a graph and let A be an Abelian group of order $k \geq 2$. Then $\chi(G) \leq k$ if and only G admits a nowhere-zero A-tension.*

Question 6

(i) Prove Proposition 9.

(ii) Explain why a nowhere-zero A-tension of $G = (V, E)$ remains a nowhere-zero A-tension of $G\backslash e$, where e is any edge of G.

(iii) Dually, show that if ϕ is a nowhere-zero A-flow of G then, with no change in its values on $E \setminus \{e\}$, it is also a nowhere-zero A-flow of G/e.

Although flows and tensions are defined relative to an orientation of G, the structure of $\mathcal{Z}_A$ and $\mathcal{K}_A$ (in particular, their size) is independent of the choice of orientation. Given an A-flow ϕ under orientation ω, by replacing $\phi(e)$ by $-\phi(e)$ for each edge e on which ω and ω differ we obtain an A-flow of G under orientation ω'. A similar observation can be made for A-tensions.

The *support* of $\phi \in \mathcal{C}_A$ is defined by $\mathrm{supp}(\phi) = \{e \in E : \phi(e) \neq 0\}$. A subset $S \subseteq E$ is a *minimal support* if $S = \mathrm{supp}(\phi)$ for some flow ϕ and the only flow whose support is properly contained in S is the zero flow. The set of A-flows with a given minimal support (together with the zero flow) form a one-dimensional space of flows, namely of the form $a\overrightarrow{\chi}_C$ for some $a \in A$ and circuit C. A *primitive* A-flow is a flow ϕ with minimal support and for which each $\phi(e)$ is 0, 1 or -1. In other words, ϕ is equal to $\pm\overrightarrow{\chi}_C$ for some circuit C. A $\mathbb{Z}$-flow π *conforms* to a $\mathbb{Z}$-flow ϕ if $\mathrm{supp}(\pi) \subseteq \mathrm{supp}(\phi)$ and $\pi(e)\phi(e) > 0$ for $e \in \mathrm{supp}(\pi)$.

Question 7

(i) Explain why for a given $\mathbb{Z}$-flow ϕ there is a primitive $\mathbb{Z}$-flow π which conforms to ϕ. Show that any $\mathbb{Z}$-flow ϕ is the sum of integer multiples of primitive $\mathbb{Z}$-flows, each of which conforms to ϕ.

(iii) Prove that if ϕ is a nowhere-zero $\mathbb{Z}_k$-flow then there is a nowhere-zero $\mathbb{Z}$-flow ψ for which $\psi(e) \equiv \phi(e) \pmod{k}$ and $-k < \psi(e) < k$.

(iv) Deduce that if G has a nowhere-zero $\mathbb{Z}_k$-flow then it has a nowhere-zero $\mathbb{Z}_{k+1}$-flow.

[*Insert here some build-up to the following "equivalence theorem".*]

Theorem 10. *Let G be a graph with an orientation of its edges. For every $k \geq 2$, the following conditions are equivalent:*

(i) There exists a nowhere-zero $\mathbb{Z}_k$-flow in G.

(ii) For any Abelian group A of order k, there exists a nowhere-zero A-flow in G.

(iii) There exists a nowhere-zero k-flow in G.

Why is the notion of a nowhere-zero $\mathbb{Z}$-flow important? Such a question is always difficult to answer, but let us at least try.

4 4CC

The Four Colour Conjecture (4CC for short) – people like it so much that they persist in calling it a conjecture even so long after it has become a theorem! – is one of the problems which have shaped graph theory into the form we know it today. And the definition of a nowhere-zero $\mathbb{Z}$-flow allows us to view colouring of planar graphs from a new perspective.

Let $G = (V, E)$ be a plane graph with set of faces $\mathcal{F}(G)$, and $G^* = (V^*, E^*)$ its dual. Recall from Chapter **??** that we can identify $\mathcal{F}(G)$ with V^* and E^* with E. Suppose further that ω is an orientation of G and that ω^* is the dual orientation of G^*.

Let $\kappa : \mathcal{F}(G) \to \{1, ..., k\}$ be an arbitrary mapping. This is the subject of 4CC: we want to find such a mapping κ where neighbouring faces get different values (in which case we shall call κ a *colouring* of the faces of G) while taking the value k to be as small as possible. You know that this is the same thing as asking for the chromatic number of the dual graph G^*. Here we shall need the oriented version of this concept.

Given a mapping $\kappa : \mathcal{F}(G) \to \{1, ..., k\}$, the mapping $D^{*T}\kappa : E^* \to \{0, \pm 1, ..., \pm k\}$ is defined by $(D^{*T}\kappa)(XY) = \kappa(Y) - \kappa(X)$, when $X \xrightarrow{\omega^*} Y$. By duality, the mapping $D^{*T}\kappa$ can be identified with the mapping $D^T\kappa$ once E has been identified with E^*, i.e., $D^T\kappa(e) = D^{*T}\kappa(e^*)$.

We have then the following:

Theorem 11. *For any plane graph G the following statements are equivalent:*

(i) κ is a colouring of the faces of G by k colors;

(ii) κ is a colouring of the vertices of the dual graph G^* by k colors;

(iii) $D^T\kappa$ is a nowhere-zero $\mathbb{Z}$-flow of G which only uses values in $\{\pm 1, \pm 2, ..., \pm(k-1)\}$.

The equivalence of (i) and (ii) follows from the definitions. Assume (ii). Let $\kappa : \mathcal{F}(G) \to \{1, 2, \ldots, k\}$ be a colouring of the faces of G (vertices of G^*). To prove (iii) it suffices to consider an arbitrary vertex x of G and to prove that $\sum_{x \in e \in E} D^T\kappa(e) = 0$. However $\sum_{x \in e \in E} D^T\kappa(e) = \sum_{x \xrightarrow{\omega} y}(\kappa(y) - \kappa(x)) + \sum_{x \xleftarrow{\omega} y}(\kappa(x) - \kappa(y))$.

Let $X_1, ..., X_t$ be all the faces which have x on their boundary, listed in clockwise order. By duality the previous sum is $\sum_{X_i \xrightarrow{\omega^*} X_{i+1}}(\kappa(X_{i+1}) - \kappa(X_i)) + \sum_{X_i \xleftarrow{\omega^*} X_{i+1}}(\kappa(X_i) - \kappa(X_{i+1})) = 0$. Thus $D^T\kappa$ is an nowhere-zero $\mathbb{Z}$-flow.

Conversely, let $\phi : E \to \mathbb{Z}$ be a nowhere-zero $\mathbb{Z}$-flow taking values in $\{\pm 1, \ldots, \pm(k-1)\}$. Fix a face X_0 and define $\kappa(X_0) = 1$. Extend the mapping κ to all faces X as follows: if $\kappa(X)$ is not yet defined, while $\kappa(Y)$ is defined, and $X \xrightarrow{\omega^*} Y$, then we put $\kappa(X) = \kappa(Y) - \phi(e)$, where e is the edge of G corresponding to XY (and with dual orientation). Similarly if $\kappa(X)$ is not yet defined, while $\kappa(Y)$ is defined, and $X \xleftarrow{\omega^*} Y$ then we put $\kappa(X) = \kappa(Y) + \phi(e)$, where e is again the edge of G corresponding to XY in G^*. This definition is correct (of course, this is where we use that ϕ is a *flow*) and as ϕ was a *nowhere-zero* $\mathbb{Z}$-flow we obtain a proper colouring of G^*. Of course some of the colours may be negative numbers. This may be corrected by starting not with $\kappa(X_0) = 1$ but with a sufficiently large positive integer.

> **Question 8**
> Why is the mapping κ correctly defined? Why is its image contained in $\{1, \ldots, k\}$?

Corollary 12. *Every planar graph is 4-colourable if and only if every planar graph has a nowhere-zero* 4*-flow.*

This is part of the duality between colouring and flows where we have a one-to-one correspondence (colourings of a planar graph, flows of its dual). But this does not restrict us (in the same way as is the case for duality in Linear Programming) to using these concepts and problems *for graphs in general.* And this leads (as we would like to demonstrate) to a very fruitful area of contemporary combinatorics.

One more question: Why do we care about other formulations of 4CC seeing that the problem has long been solved? Some may find a reason lies in the fact that all known proofs of 4CC still do not have a satisfactory air about them; see [2, 16, 19, 15]. But another reason (and we think it is a more important one) is that the relevance of this problem is so vast that just to have a new (let us say essentially new) reformulation of 4CC is welcome and studied intensively. New equivalences are a rare article. See [9] and [7] for spectacular additions to the list.

5 Duality of bases for A-tensions and A-flows

For a connected graph we have seen the signed characteristic vectors of cocircuits are spanned by the linearly independent set of vectors $\{\overrightarrow{\chi}_{\omega^+(\{u\})\cup\omega^-(\{u\})} : u \in V \setminus \{v\}\}$, where v is an arbitrary vertex.

A pair of bases, one for cocircuits and the other for circuits, can be defined relative to a fixed spanning tree of the graph. These bases are, in a sense we shall make precise shortly, dual to each other.

Proposition 13. *Let G be a connected graph, D its incidence matrix (for some orientation of G), and T a spanning tree of G.*

The signed characteristic vectors of the circuits $\{C_{T,e} : e \in E \setminus T\}$ form a basis for the set of A-flows of G. The signed characteristic vectors of the cocircuits $\{K_{T,e} : e \in E\}$ form a basis for the space of A-tensions of G.

Proof. A given edge $e \in E \setminus T$ belongs to $C_{T,e}$ but no other cycle $C_{T,f}$ for $f \neq e$. Hence the signed characteristic vectors $\{\overrightarrow{\chi}_{C_{T,e}} : e \in E \setminus T\}$ are linearly independent, and form a basis since there are $|E \setminus T| = n(G)$ of them.

Likewise, a given edge $e \in T$ belongs to $K_{T,e}$ but to no other $K_{T,f}$ for $f \neq e$, so the $|T| = r(G)$ signed indicator vectors of these cocircuits are linearly independent. □

We now come to an abstract expression of the fact that we have already encountered that A-tensions of a planar graph correspond to A-flows of its dual:

Proposition 14. *Let G be a connected plane graph with orientation ω and G^* its dual graph with dual orientation ω^*. Let D denote the incidence matrix of G^ω and D^* the incidence matrix of $(G^*)^{\omega^*}$. Then $D^*D^T = O$. Also,* $\ker(D^*) = \mathrm{im}(D^T)$ *and* $\mathrm{im}((D^*)^T) = \ker(D)$.

Proof. Given a vertex $v \in V$ and face X incident with v, there are exactly two edges e, f belonging to X and with v as an endpoint. Then

$$(D^*D^T)_{X,v} = (D^*)_{X,e}(D)_{v,e} + (D^*)_{X,f}(D)_{v,f}. \tag{1}$$

Note that reversing the orientation of edge e does not change the value of $(D^*)_{X,e}(D)_{v,e}$ since both signs are flipped. Likwise for reversing the orientation of e. Taking the orientation that directs e into v and f out of v (for example), we calculate that (1) is equal to $(+1)(+1) + (+1)(-1) = 0$. Hence $D^*D^T = O$, so that $\mathrm{im}(D^*)$ is orthogonal to $\mathrm{im}(D^T)$. Since D has rank $r(G)$ and D^* has rank $r(G^*) = n(G)$ it follows that $\mathrm{im}((D^*)^T) = \ker(D)$ and $\ker(D^*) = \mathrm{im}(D^T)$. □

Thus we have it formalized in stone what we already by now know: A-tensions of G are precisely A-flows of G^*. Moreover A-tensions of G with minimal support are A-flows of G^* with minimal support. In particular, circuits of G^* are cocircuits of G, and cocircuits of G^* are circuits of G. We learned in Chapter **??** that this is a defining property of planar graphs, that the dual of the cycle matroid of G is also a graphic matroid, namely the cycle matroid of the planar dual graph G^* (Theorem **??**).

Since faces of G correspond to vertices of G^*, another natural basis for circuits of a connected plane graph G consists of the characteristic vectors of all but one of the face boundaries (say all but the outer face). This corresponds to the cocircuit basis of G^* obtained by taking the characteristic vectors of the edges incident with a common vertex, for all but one vertex of G^*.

Call a graph G^* the *abstract dual* of a graph G if $E(G) = E(G^*)$ and the cocircuits of G^* are precisely the circuits of G. This is to say that the cutset space of G^* is the cycle space of G: the cycle matroids of G and G^* are dual. We have seen that a connected planar graph has an abstract dual, equal to its geometric dual when it is embedded in the plane. This is a defining property of planar graphs:

Theorem 15. (Whitney, 1933) *A graph is planar if and only if it has an abstract dual.*

For a proof see for example [6, ch. 4].

6 Examples of nowhere-zero flows

We saw earlier in Proposition 8 that if ϕ is a $\mathbb{Z}_2$-flow of a graph G then the support of ϕ (the set of edges where it is non-zero) is an edge-disjoint union of circuits. The reader is invited to deduce the following corollary:

Proposition 16. *The faces of a plane graph can be properly coloured with two colours if and only if all the vertices have even degree.*

Nowhere-zero $\mathbb{Z}_3$-flows are in general difficult customers (we shall later encounter a longstanding conjecture of Tutte concerning them), but by restricting attention to 3-regular graphs things become easier:

Proposition 17. *A cubic graph G has a nowhere-zero $\mathbb{Z}_3$-flow if and only if it is bipartite.*

Proof. Given a nowhere-zero $\mathbb{Z}_3$-flow of G, choose the orientation of G so that the value on each edge is $+1$. Then in this orientation every vertex is either a source or sink and this yields a proper vertex 2-colouring of G. Conversely, if G has a proper 2-colouring κ with colours $0, 1 \in \mathbb{Z}_3$ then, directing vertices coloured 0 towards vertices coloured 1, the potential difference $\delta\kappa$ is equal to 1 everywhere and so is not only a nowhere-zero $\mathbb{Z}_3$-tension but also a nowhere-zero $\mathbb{Z}_3$-flow, since G is cubic. □

When translated to planar graphs this gives a theorem of Heawood from 1890:

Proposition 18. *A plane triangulation G has a proper vertex 3-colouring if and only if it has a proper face 2-colouring (equivalently, G is Eulerian).*

Question 9
Prove Proposition 18.

Thus we have found examples of graphs with a nowhere-zero 3-flow. What about a nowhere-zero 4-flow? Let us try to give some examples. You have probably heard of this one:

Proposition 19. *A simple cubic planar graph has a edge 3-colouring if and only if its faces can be properly coloured with four colours.*

(A graph G is said to be edge k-colourable if we can colour the edges of G with k colours such that any two incident edges have different colours.)

This is Tait's theorem (from 1880)[18] which was isolated in order to give one of the first proofs of 4CC. It also led to study of *Hamiltonian graphs* and to the *Petersen graph*. This text wouldn't be complete without its picture.

The same argument that Tait used to prove Proposition 19 can be generalized to non-planar graphs:

Proposition 20.
A cubic graph G has a nowhere-zero 4-flow if and only if it is has a proper edge 3-colouring.

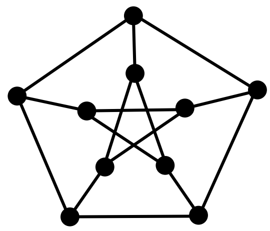

Figure 2: Petersen Graph

By Theorem 10 a graph has a nowhere-zero 4-flow if and only if it has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Let the non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ be a, b, c. We have $a + b + c = 0$ and $a + a = b + b = c + c = 0$. From this it is easy to see that a mapping $f : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow if and only if it is a proper edge 3-colouring using the colours a, b, c.

Question 10 Using the equivalence of Proposition 20, show that the Petersen graph does not have a nowhere-zero 4-flow. (Hint: consider edges of a fixed colour in a putative edge 3-colouring, at least one of which must occur on the outer 5-cycle in Figure 2. What does this imply about the number of occurrences of this colour on the inner 5-cycle?)

The Petersen graph does however have nowhere-zero 5-flows, as shown in Figure 3.

The following is an alternative characterization of graphs with a nowhere-zero 4-flow:

Proposition 21. *A graph $G = (V, E)$ has a nowhere-zero 4-flow if and only if $E = E_1 \cup E_2$ and each of the graphs (V, E_1) and (V, E_2) is Eulerian.*

By Theorem 10 has a nowhere-zero 4-flow if and only if it has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow ϕ. Write $\phi = (\phi_1, \phi_2)$ and observe that the ϕ_i's are $\mathbb{Z}_2$-flows which are nowhere-zero $\mathbb{Z}_2$-flows on the support E_i of ϕ_i. However, as we observed at the beginning of this section, this happens if and only if the subgraph on edge set E_i has all vertex degrees even. Moreover ϕ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow if and only if $E = E_1 \cup E_2$.

Proposition 18 gives a nowhere-zero $\mathbb{Z}_2$-flow condition for a plane triangulation to have a proper 3-colouring of its vertices (a nowhere-zero $\mathbb{Z}_3$-tension). Underlying this is the dual version of Proposition 17.

Using Proposition 21 and tension-flow duality, a planar graph has a proper 4-colouring of its vertices if and only if its dual is the union of two if its Eulerian subgraphs. This criterion for a graph to have a nowhere-zero 4-flow emerges by using the fact that a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow is supported in each component on an Eulerian subgraph. If we had considered nowhere-zero $\mathbb{Z}_4$-flows rather than $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows then what criterion would we obtain instead? For a cubic graph

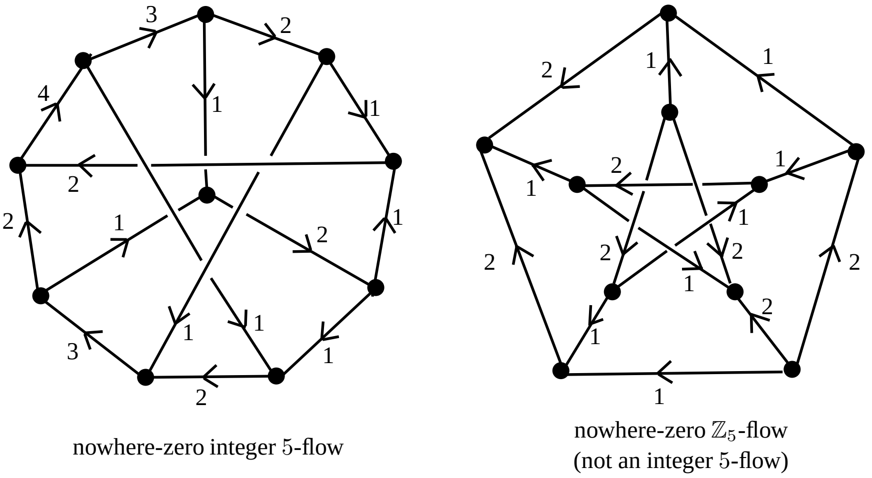

Figure 3: The Petersen graph (in two of its guises) with on the left a nowhere-zero 5-flow (also a nowhere-zero $\mathbb{Z}_5$-flow) and on the right a nowhere-zero $\mathbb{Z}_5$-flow.

we would find a perfect matching (edges receiving the value 2) together with a collection of oriented circuits (edges with value ± 1), each of which has the property that relative to the circuit orientation the values assigned to its edges alternate between 1 and -1 (i.e., the circuit is even and edge 2-coloured). This is effectively Proposition 20, which gives an edge 3-colouring equivalent to the existence of a nowhere-zero 4-flow of a cubic graph.

Question 11
Characterize graphs G that have a nowhere-zero $\mathbb{Z}_2^r$-flow in terms of Eulerian subgraphs. What is the dual version: when does a graph have a nowhere-zero $\mathbb{Z}_2^r$-tension?

An Eulerian orientation of a graph G is an orientation of G with the property that the indegree at a vertex is equal to its outdegree. Clearly G must be Eulerian, and by decomposing G into an edge-disjoint union of cycles there exist Eulerian orientations of G in this case.

Proposition 22. *Let G be a 4-regular graph. Then there is a one-to-one correspondence between nowhere-zero $\mathbb{Z}_3$-flows of G and Eulerian orientations of G.*

Proof. For a given nowhere-zero $\mathbb{Z}_3$-flow of G, arrange the orientation σ of G so that each flow value is equal to 1. Then the only way to obtain net flow zero at a vertex is to have two edges directed out and two edges directed in. In other words, the orientation σ is Eulerian. (Put alternatively, keep the fixed orientation σ of G and for a given nowhere-zero $\mathbb{Z}_3$-flow of G preserve the orientation when flow value is $+1$ and reverse the orientation when flow value is -1: the result is an Eulerian orientation, uniquely defined by the flow values and σ.) □

In Chapter **??** we shall return to the topic of Eulerian orientations. Nowhere-zero $\mathbb{Z}_3$-flows of a graph G more generally correspond to orientations of G in which every vertex has indegree congruent to outdegree modulo 3. See Chapter **??** for Tutte's still open conjecture as to which graphs have a nowhere-zero $\mathbb{Z}_3$-flow.

We move on now to nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows, whose significance in the history of attempts at proving the Four Colour Theorem we shall briefly describe. First a lemma which is not only of immediate use, but also to the problem of counting nowhere-zero A-flows that we consider in the next section. Let $G = (V, E)$ be a connected graph and T a spanning tree of G. Let A be an Abelian group and $\phi_0 : E \setminus T \to A$. Then there is a unique A-flow ϕ of G such that $\phi(e) = \phi_0(e)$ for $e \in E \setminus T$.

Proof. The vector

$$\phi = \sum_{e \in E \setminus T} \phi_0(e) \overrightarrow{\chi}_{C_{T,e}}$$

as a linear combination of basis vectors for $\mathcal{C}_A$ is an A-flow and since $e \notin C_{T,f}$ when $f \neq e$ the value of ϕ at e is given by $\phi(e) = \phi_0(e)$. Conversely, if an A-flow takes value $\phi_0(e)$ at each $e \in E \setminus T$ then it is equal to ϕ as defined above, since any vector in $\mathcal{C}_A$ has a unique expression as a linear combination of basis vectors. □

Theorem 23. *A graph with a Hamiltonian circuit (a circuit traversing all vertices of G) has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.*

Proof. Let H be a Hamitonian circuit of G and T a spanning tree (a path) obtained from H by removing one of its edges. Let ϕ_1 be a $\mathbb{Z}_2$-flow of G with support containing $E \setminus T$, which exists by Lemma 6 (taking $\phi_0(e) = 1$ for $e \in E \setminus T$). Let ϕ_2 be the $\mathbb{Z}_2$-flow with support the circuit H. Then (ϕ_1, ϕ_2) is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow of G. □

In the early years of trying to prove 4CC, Tait conjectured in 1884 that every 3-connected planar graph was Hamiltonian (an example of 2-connected planar non-Hamiltonian was known, consisting of 20 vertices and 12 pentagonal faces). Tutte in 1956 gave a counterexample with 46 vertices. (See e.g. [17] for diagrams and a succinct historical account of variations on 4CC.) See also the Herschel graph depicted in Figure XX in Chapter **??**.

We have found many graphs that have a nowhere-zero A-flow when $|A| \leq 4$. In the dual problem, no matter how large we choose $|A|$ there will always be graphs that do not have a nowhere-zero A-tension, namely those graphs with chromatic number exceeding $|A|$. The simplest obstruction to a proper k-colouring is an induced clique on $k+1$ vertices. Is there an obstruction to a nowhere-zero A-flow when $|A| \geq 5$? Certainly not cliques, as we shall see shortly. But which graphs do not have a nowhere-zero $\mathbb{Z}_5$-flow? Tutte (again!) had thoughts upon this matter, as we shall see in Chapter **??**, and no one has yet resolved the question. But let us keep to simple things for the moment and see off the complete graphs as being quite tame creatures when it comes to nowhere-zero flows.

The complete graph K_2 is a bridge and therefore does not have a nowhere-zero flow. K_3 is Eulerian and so has a nowhere-zero $\mathbb{Z}_2$-flow. K_4 has a proper edge 3-colouring and hence has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. On the other hand, K_4 does not have a nowhere-zero $\mathbb{Z}_3$-flow since it is a non-bipartite cubic graph and does not have a nowhere-zero $\mathbb{Z}_2$-flow since it is not Eulerian.

Proposition 24. *K_n has a nowhere-zero $\mathbb{Z}_2$-flow when $n \geq 3$ is odd. K_n has a nowhere-zero $\mathbb{Z}_3$-flow when $n \geq 6$ is even.*

Proof. The case of odd n follows since K_n is Eulerian. For $n = 6$ we have K_6 is the edge-disjoint union of two copies of K_3 and one copy of $K_{3,3}$. Each of these graphs has a nowhere-zero $\mathbb{Z}_3$-flow($K_{3,3}$ since it is a cubic biparitite graph). The union of these flows makes a nowhere-zero $\mathbb{Z}_3$-flow of K_6.

Consider now even $n > 6$ and assume the assertion of the theorem holds for $n-2$. The graph K_n is the edge-disjoint union of K_{n-2} and $K^+_{2,n}$, where the latter is $K_{2,n}$ with an edge e added between the vertices of degree n. By hypothesis K_{n-2} has a nowhere-zero $\mathbb{Z}_3$-flow. To make a nowhere-zero $\mathbb{Z}_3$-flow of $K^+_{2,n}$ take the sum of nowhere-zero $\mathbb{Z}_3$ flows on each of the n triangles: this is non-zero on all but possibly the edge e. If necessary, make the value on e non-zero by adding in the flow again from a single (aribitrary) triangle of edges e, e_1, e_2: this makes the value on e non-zero, and reverses the sign of the flow on e_1 and e_2. We have thus constructed a nowhere-zero $\mathbb{Z}_3$-flow of K_n. □

7 The flow polynomial

We turn to the problem of *counting* nowhere-zero A-flows. (The dual problem of counting nowhere-zero A-tensions is the subject of the next chapter.)

Theorem 25. (Tutte, [20].) *Let A be a finite Abelian group of order k and G a graph with an orientation of its edges. Then the number of nowhere-zero A-flows of G is*

$$F(G;k) = \sum_{F \subseteq E} (-1)^{|E|-|F|} k^{n(F)}.$$

Proof. By Lemma 6 the number of A-flows of any subgraph (V, F) of $G = (V, E)$ is equal to $k^{|F|-r(F)}$, since a maximal spanning forest of (V, F) has $r(F)$ edges. Equivalently, $k^{n(F)}$ is the number of A-flows of G whose support is contained in F. The result follows by the inclusion-exclusion principle. □

The polynomial $F(G; k)$ is called the *flow polynomial* of G. Theorem 25 implies that the number of nowhere-zero A-flows depends only on $|A|$, not on the structure of A as a group. In particular, the existence of an A-flow only depends on $|A|$, i.e., if A and A' are Abelian groups with $|A| = |A'|$ then G has a nowhere-zero A flow if and only if G has a nowhere-zero A'-flow. As a consequence, the existence of a nowhere-zero A-flow implies the existence of a nowhere-zero A'-flow when $|A'| > |A|$. This is because (as Tutte first showed in 1950 – see e.g. [8], [14], [6] for details – and you have already shown in Question X above) a nowhere-zero k-flow exists if and only if a nowhere-zero $\mathbb{Z}_k$-flow exists, whence if $k' > k$ then there is a nowhere-zero $\mathbb{Z}_{k'}$-flow whenever there is a nowhere-zero $\mathbb{Z}_k$-flow. (Thinking of A-flows as duals of A-tensions, it is obvious that if G has a nowhere-zero A-tension then it has a nowhere-zero A'-tension, by using the correspondence of nowhere-zero A-tensions with proper A-colourings.)

Proposition 26. *The flow polynomial satisfies*

$$F(G;k) = \begin{cases} F(G/e;k) - F(G\backslash e;k) & e \text{ ordinary,} \\ 0 & e \text{ a bridge,} \\ (k-1)F(G\backslash e) & e \text{ a loop,} \\ 1 & E = \emptyset. \end{cases}$$

Proof. When $E = \emptyset$ the subgraph expansion for $F(G;k)$ gives $F(G;k) = 1$. When G has a bridge e it does not have a nowhere-zero flow, for $\{e\}$ is a cut of G. If e is a loop, on the other hand, then we can freely assign any non-zero value to it and still have a nowhere-zero flow. When e is ordinary, we have a bijection between nowhere-zero flows of $G\backslash e$ and flows of G that are zero only at e, and between nowhere-zero flows of G/e and flows of G that are nowhere-zero except possibly at e. (This argument also works when e is a bridge, but it needs to be shown that in this case $F(G\backslash e;k) = F(G/e;k)$, which amounts to showing that $F(G;k) = 0$.) □

Question 12
Suppose $G = (V,E)$ is a connected graph and A a finite Abelian group of order k.

(i) Given a spanning tree T and $\theta_0 : T \to A$, prove there is a unique A-tension θ of G such that $\theta(e) = \theta_0(e)$.

(ii) Deduce that the number of nowhere-zero A-tensions of G is given by

$$F^*(G;k) = \sum_{F \subseteq E} (-1)^{|E|-|F|} k^{r(F)}.$$

(iii) Formulate and prove a deletion-contraction recurrence satisfied by the polynomial $F^*(G;k)$.

Kochol [10] shows that the number of nowhere-zero k-flows is also a polynomial in k (*not* the same as the flow polynomial $F(G;k)$) - this polynomial counting integer flows does not satisfy a deletion-contraction recurrence.

For any finite Abelian group A there are loopless graphs G that do not have a nowhere-zero A-tension (take G with $\chi(G) > |A|$). The situation for nowhere-zero A-flows is quite different, where bridges are the only obstruction to having a nowhere-zero A-flow once $|A|$ is sufficiently large. In fact Seymour showed that $|A| \geq 6$ will do, and it is a famous conjecture of Tutte that in fact $|A| \geq 5$ suffices. Within the class of planar graphs – and as Whitney showed (see Theorem 15) this class is precisely the set of graphs closed under duality – 4CC tells us that we do have a symmetric situation: if $|A| \geq 4$ then any planar graph has a nowhere-zero A-tension and a nowhere-zero A-flow. It is when we move out of the class of planar graphs that a fundamental difference between the dual notions of flows and tensions arises. Of course within the more general world-view of matroids this asymmetry disappears (there are regular matroids with arbitrarily large flow number, as well as with arbitrarily large chromatic number).

But we shall wait until Chapter 8 to take up this story again. First we shall continue the theme of counting nowhere-zero A-flows and nowhere-zero A-tensions, for the latter brings us to an historically very important graph invariant: the chromatic polynomial.

8 Colourings and flows in the ice model

We finish this chapter with a charming illustration of how tensions and flows feature in models of physical processes. This is to but scratch the surface of an extensive literature on the appplication of combinatorics to physics – we shall meet another example in Chapter **??**.

Square ice consists of an $n \times n$ lattice arrangement of oxygen atoms. Between any two adjacent O-atoms lies one hydrogen atom, and there are also H-atoms at the left and right boundaries. The problem is to count all possible configurations in which evey O-atom is attached to exactly two of its surrounding H-atoms, forming H_2O.

There is a bijection between $n \times n$ ice configurations and Eulerian orientations on the lattice graph of O-atoms, with boundary conditions. Let u and v be two adjacent O-atoms. Orient the edge $u \longrightarrow v$ if the H-atom between u and v is attached to v. On the left and right boundaries all edges are incoming (each H-atom on the boundary is attached to an O-atom horizontally). On the top and bottom boundaries all edges are outgoing. See Figure 4.

In this way we get an Eulerian orientation of the $n \times n$ lattice graph with hanging boundary edges (each missing one endpoint).

The number of ice configurations is the number of Eulerian orientations of the $n \times n$ lattice graph with boundary conditions (incoming edges left and right, outgoing edges top and bottom). Each O-atom has six possible attachments to neighbouring H-atoms, corresponding to the six possible orientations at a vertex of degree 4 with two incoming and two outgoing edges. (This gives the alternative name of "six-vertex model" for the ice model.)

The $n \times n$ lattice graph of O-atoms with directed edges added as described

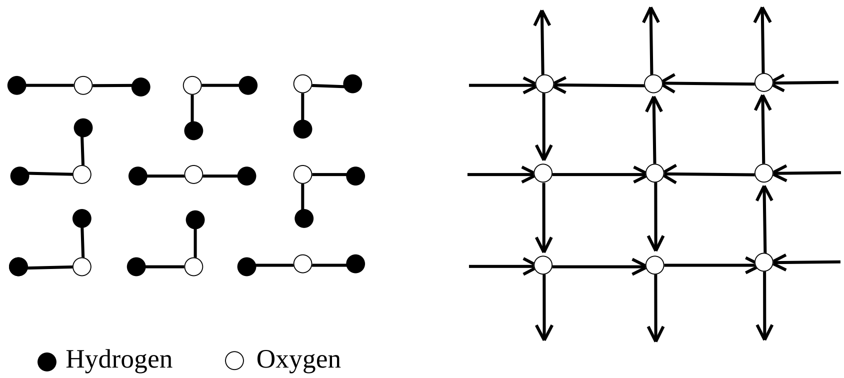

Figure 4: A configuration in the 3×3 square ice model, and its associated orientation.

gives an $(n+1) \times (n+1)$ array of square cells, where each O-vertex is incident with four cells. The cells can be $\mathbb{Z}_3$-coloured by the following rule. Colour the top left corner 0. Suppose a and b are neighbouring cells such that the edge that separates them has orientation having a to the left and b to the right, and that a and b have colours $c(a)$ and $c(b)$ respectively. Then $c(b) = c(a) + 1$. In other words add one modulo 3 going from left to right across a directed edge. The boundary colours appear in sequence $0, 1, 2, 0, \ldots$, with the bottom right corner coloured 0 like the top left. (The sequence along the top is the mirror image of that along the bottom, and likewise for left and right boundaries.)

This gives a bijection between $n \times n$ ice configurations and proper $\mathbb{Z}_3$-colourings of the $(n+1) \times (n+1)$-array of cells, observing the boundary conditions.

An alternative way to see this 3-colouring procedure is to first add edges to the $n \times n$ lattice graph $L_{n,n}$ to make it a 4-regular graph as follows. Given $L_{n,n}$ on vertex set $[n] \times [n]$, add edges between $(i,1)$ and $(1,i)$ for each $i \in [n]$ and edges between (i,n) and (n,i) for each $i \in [n]$. This yields a 4-regular planar graph $\widetilde{L}_{n,n}$ (with loops at the two corners $(1,1)$ and (n,n)). An Eulerian orientation of $\widetilde{L}_{n,n}$ is obtained by the same rule of directing O-atom u towards O-atom v when v is attached to the H-atom between u and v, the orientation of edges joining boundary O-atoms being determined by always directing edge into those vertices on the left or right boundaries. By tension-flow duality, each nowhere-zero $\mathbb{Z}_3$-flow (Eulerian orientation) of $\widetilde{L}_{n,n}$ corresponds to a nowhere-zero $\mathbb{Z}_3$ tension of the dual graph $\widetilde{L}^*_{n,n}$, i.e. to three proper $\mathbb{Z}_3$-colourings of the faces of $\widetilde{L}_{n,n}$. Fixing the colour of either of the loop faces to be 0, it is easy to see that this corresponds to the cell-colouring described above. See Figure 5.

This 3-coloured version of the square ice problem is the starting point for the proof of the remarkable formula obtained by Zeilberger and Kuperberg in

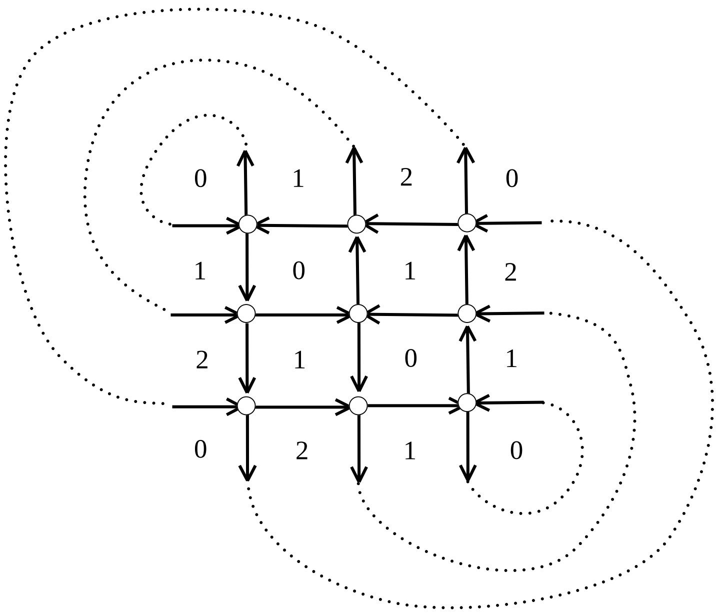

Figure 5: Eulerian orientation of 4-regular graph corresponds to a nowhere-zero $\mathbb{Z}_3$-flow, whose dual is a nowhere-zero $\mathbb{Z}_3$-tension, from which we get a proper face 3-colouring.

1996: the number of $n \times n$ ice configurations is equal to

$$\frac{(3n-2)!(3n-5)!\cdots 4!1!}{(2n-1)!(2n-2)!\cdots(n+1)!n!}.$$

See [1, Chapter 10] and [5].

In the general case, an ice model concerns the number of ways of orienting a 4-regular graph G such that each vertex has 2 incoming edges and 2 outgoing edges, i.e., an Eulerian orientation of G.

In Proposition 22 we saw that Eulerian orientations of a 4-regular graph correspond to nowhere-zero $\mathbb{Z}_3$-flows of G, so that there are $F(G;3)$ ice configurations on G.

Although finding an Eulerian orientation can be done polynomial time, in general computing the number of them is #P-complete, as proved by Mihail and Winkler [13]. In other words, computing $F(G;3)$ is #P-complete even on the class of 4-regular graphs. (In Chapter **??** we shall have another look at the graph parameter counting the number of Eulerian orientations of a not necessarily 4-regular graph.)

Proposition 27. *Let $G = (V,E)$ be a 4-regular graph. Then $F(G;3) \geq \left(\frac{3}{2}\right)^{|V|}$.*

Proof. Use induction on the number of vertices of G. The case of a single vertex with two loops has $F(G;3) = 4 \geq \frac{3}{2}$.

For a graph on n vertices, choose one, say v, and partition Eulerian orientations of G according to which of the six possible configurations is at v. Fix an Eulerian orientation of G. Let a, b, c, d be the neighbours of v and suppose that $a \longrightarrow v$, $b \longrightarrow v$, $v \longrightarrow c$, $v \longrightarrow d$.

Define a 2-in 2-out digraph G_1 on vertex set $V \setminus \{v\}$ as follows. Take the same edge orientations as G for edges not incident with v, together with directed edges $a \longrightarrow c$, $b \longrightarrow d$ to replace the four edges of G incident with v. Similarly, define the 2-in 2-out digraph G_2 by in a similar way except taking directed edges $a \longrightarrow d$ and $b \longrightarrow c$.

According to the configuration of oriented edges incident with v the resulting digraphs G_1 and G_2 have each one of the three types of "transition" at v, as illustrated in Figure 6 below.

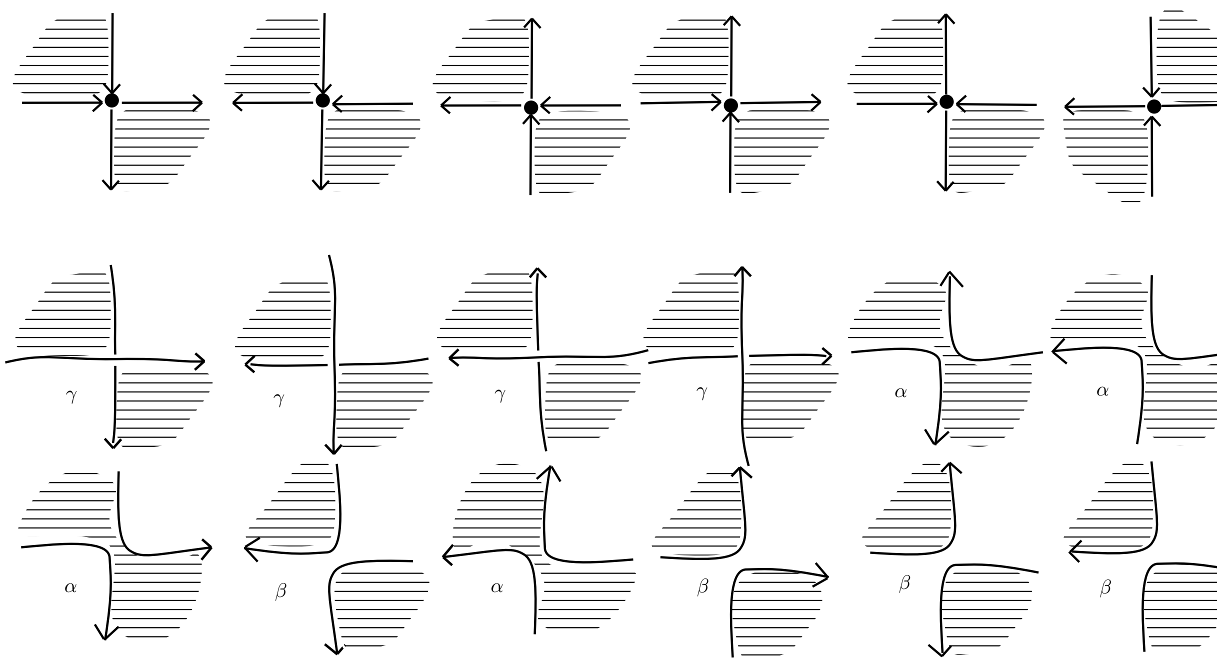

Figure 6: Two possible transitions at a vertex for each of the six configurations of orientations of its four incident edges (given in the top row). Three types of transition: white (α), black (β) and crossing (γ).

Depending on which of the six possible configurations of directed edges is at v, the digraphs G_1 and G_2 are Eulerian orientations of two of three possible 4-regular graphs $G_\alpha, G_\beta, G_\gamma$, according as the transition type at v is white (α), black (β) or crossing (γ). See Figure 7 below.

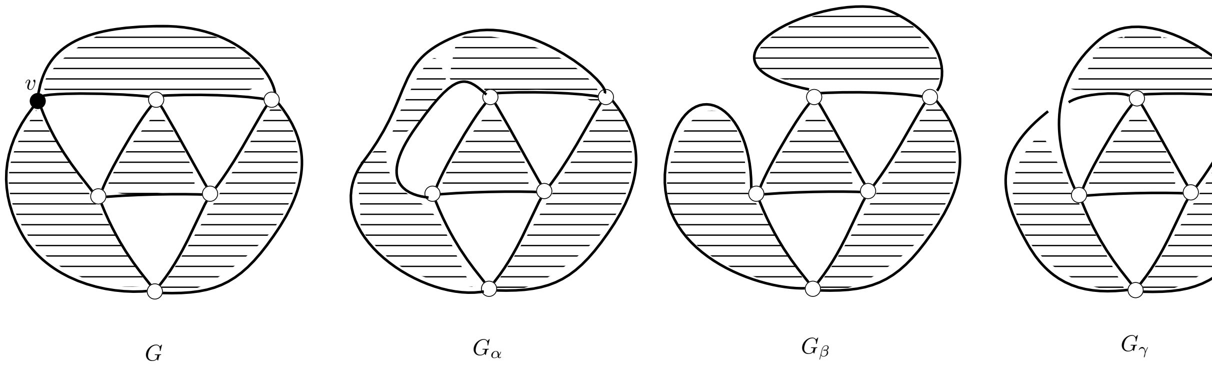

Figure 7: The three possible types of transition (black, white, crossing) at a vertex v of a 4-regular graph G. Which transition occurs depends on how the oriented edges incident with v are "tied together" when eliminating v from G to obtain either of the two 4-regular graphs G_1 or G_2.

In Figure 7 notice that the transition type α occurs four times, with all four possible configurations of orientations of the two edges. A similar observation holds for the transition types β and γ.

Therefore, by considering the two possible ways to "tie together" two edges with matching directions in all six configurations of orientations of edges incident with v, we find that

$$F(G_\alpha;3) + F(G_\beta;3) + F(G_\gamma;3) \leq 2F(G;3),$$

and by induction hypothesis

$$3 \cdot \left(\frac{3}{2}\right)^{n-1} \leq 2F(G;3),$$

yielding the desired lower bound. □

In the square ice model we take $G \cong \widetilde{L}_{n,n}$ the $n \times n$ grid with edges added between $(i,1)$ and $(1,i)$ and edges between (i,n) and (n,i), for each $i \in [n]$.

Lieb proved in 1967 that for the square lattice

$$\lim_{n\to\infty} F(\widetilde{L}_{n,n};3)^{\frac{1}{n^2}} = \left(\frac{4}{3}\right)^{\frac{3}{2}} \approx 1.5396.$$

This is quite close to the lower bound of $\frac{3}{2}$ given by Proposition 27.

Suppose for a moment that G is the medial graph of a cubic planar graph H. Then $P(G;3)$ is the number of proper edge 3-colourings of H, so if we had a positive lower bound for $F^*(G;3)$ rather than $F(G;3)$ we would have a quantative version of 4CC: bounding the number of proper edge 3-colourings of a cubic planar graph H from below positively would yield a lower bound on the number of proper face 4-colourings of H (why?). Needless to say such a lower bound on $F(G^*; 3)$ is not forthcoming.

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