Graph invariants, homomorphisms, and the Tutte polynomial

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February 26, 2013

1 The chromatic polynomial

1.1 The chromatic polynomial and proper colourings

There are various ways to define the chromatic polynomial $P(G;z)$ of a graph G. Perhaps the first that springs to mind is to define it to be the graph invariant $P(G;k)$ with the property that when k is a positive integer $P(G;k)$ is the number of colourings of the vertices of G with k or fewer colours such that adjacent vertices receive different colours. One then has to prove that $P(G;k)$ is indeed a polynomial in k. This can be done for example by an inclusion-exclusion argument, or by establishing that $P(G;k)$ satisfies a deletion-contraction recurrence and using induction.

However, we shall take an alternative approach and define a polynomial $P(G;z)$ by specifying its coefficients as graph invariants that count what are called colour-partitions of the vertex set of G. It immediately emerges that $P(G;k)$ does indeed count the proper vertex k-colourings of G. A further aspect of this approach is that we choose a basis different to the usual basis $\{1, z, z^2, \ldots\}$ for polynomials in z. This basis, $\{1, z, z(z-1), \ldots\}$, has the advantage that we are able to calculate the chromatic polynomial very easily for many graphs, such as complete multipartite graphs.

In this chapter we develop some of the many properties of the chromatic polynomial, which has received intensive study ever since Birkhoff introduced it in 1912 [2], perhaps with an analytic approach to 4CC in mind. Although such an approach has not led to such a proof of 4CC being found, study of the chromatic polynomial has led to many advances in graph theory that might not otherwise have ben made. In the context of this book, the chromatic polynomial played a significant role historically in Tutte's elucidation of tension-flow duality. (In the next chapter we look at Tutte's eponymous polynomial, introduced as simultaneous generalization of the chromatic and flow polynomials.)

More about graph colourings can be found in e.g. [4, ch. V], [6, ch. 5], and more about the chromatic polynomial in e.g. [1, ch. 9] and [9].

We approach the chromatic polynomial via the key property that vertices of the same colour in a proper colouring of G form an independent (stable) set in G.

Definition 1. *A* colour-partition *of a graph $G = (V,E)$ is a partition of V into disjoint non-empty subsets, $V = V_1 \cup V_2 \cup \cdots \cup V_k$, such that the colour-class V_i is an independent set of vertices in G, for each $1 \leq i \leq k$ (i.e., each induced subgraph $G[V_i]$ has no edges).*

The chromatic number *$\chi(G)$ is the least natural number k for which such a partition is possible.*

If G has a loop then it has no colour-partitions. Adding or removing edges in parallel to a given edge makes no difference to what counts as a colour-partition, since its definition depends only on whether vertices are adjacent or not.

We denote the falling factorial $z(z-1)\cdots(z-i+1)$ by $z^{\underline{i}}$.

Definition 2. *Let $G = (V,E)$ be a graph and let $a_i(G)$ denote the number of colour-partitions of G into i colour-classes. The* chromatic polynomial *of G is defined by*

$$P(G;z) = \sum_{i=1}^{|V|} a_i(G) z^{\underline{i}}.$$

For example, when G is the complete graph on n vertices,

$$P(K_n; z) = z^{\underline{n}} = z(z-1)\cdots(z-n+1),$$

with $a_i(K_n) = 0$ for $1 \le i \le n-1$ and $a_n(K_n) = 1$.

If G has n vertices then $a_n(G) = 1$ so that $P(G; z)$ has leading coefficient 1. The constant term $P(G; 0)$ is zero since z is a factor of $z^{\underline{i}}$ for each $1 \le i \le n$. If E is non-empty then $P(G; 1) = 0$, so that $z-1$ is a factor of $P(G; z)$. More generally, the integers $0, 1, \dots, \chi(G)-1$ are all roots of $P(G; z)$, and $\chi(G)$ is the first positive integer that is not a root of $P(G; z)$.

Proposition 3. *If $G = (V, E)$ is a simple graph on n vertices and m edges then the coefficient of z^{n-1} in $P(G; z)$ is equal to $-m$.*

Proof. A partition of n vertices into $n-1$ subsets necessarily consists of $n-2$ singletons and one pair of vertices $\{u, v\}$. This is a colour-partition if and only if $uv \notin E$. Hence $a_{n-1}(G) = \binom{n}{2} - m$, where m is the number of pairs of adjacent vertices, equal to the number of edges of G when there are no parallel edges. Then

$$[z^{n-1}]P(G; z) = -(1 + 2 + \cdots + n-1)a_n(G) + a_{n-1}(G) = -m.$$

□

The *join* $G_1 + G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \cup V_2$ and edge set

$$E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}.$$

For example the join of two cocliques $\overline{K}_r + \overline{K}_s$ is a complete bipartite graph $K_{r,s}$.

Proposition 4. *The chromatic polynomial of the join $G_1 + G_2$ is given by*

$$P(G_1 + G_2; z) = P(G_1; z) \circ P(G_2; z),$$

where the $\circ$ operation is defined by $z^{\underline{i}} \circ z^{\underline{j}} = z^{\underline{i+j}}$, extended linearly to polynomials.

Proof. The number of colour-partitions of $G = G_1 + G_2$ is given by

$$a_k(G) = \sum_{i+j=k} a_i(G_1)a_j(G_2),$$

since every vertex of G_1 is adjacent in G to every vertex of G_2, so that any colour-class of vertices in G is either a colour class of G_1 or a colour class of G_2. □

The operation $\circ$ treats falling factorials $z^{\underline{i}}$ as though they were usual powers z^i when multiplying together the polynomials $\sum_i a_i(G_1)z^{\underline{i}}$ and $\sum_j a_j(G_2)z^{\underline{j}}$. This is part the shadowy world of "umbral calculus"...

Question 1

(i) Find the chromatic polynomial of the wheel $C_n + K_1$ on $n+1$ vertices.

(ii) Find an expression for the chromatic polynomial of the complete bipartite graph $K_{r,s}$ relative to the factorial basis $\{z^{\underline{n}}\}$ (leaving your answer in the form of a double sum).

Definition 5. *A* proper k-colouring *of the vertices of $G = (V, E)$ is a function $f : V \to [k]$ with the property that $f(u) \neq f(v)$ whenever $uv \in E$.*

Note that the vertices of a graph are regarded as labelled and colours are distinguished: colourings are different even if equivalent up to an automorphism of G or a permutation of the colour set.

Proposition 6. *If $k \in \mathbb{N}$ then $P(G; k)$ is the number of proper vertex k-colourings of G.*

Proof. To every proper colouring in which exactly i colours are used there corresponds a colour-partition into i colour classes. Conversely, given a colour-partition into i classes there are $k^{\underline{i}}$ ways to assign colours to them. Hence the number of proper k-colourings is $\sum a_i(G)k^{\underline{i}} = P(G;k)$. $\square$

The fact that the polynomial $P(G;z)$ can be interpolated from its evaluations at positive integers gives a method of proving identities satisfied by $P(G;z)$ generally. Namely, check the truth of the identity when $z = k \in \mathbb{N}$ by verifying a combinatorial property of proper k-colourings. We finish this section with some examples.

Proposition 7. *Suppose G' is obtained from G by joining a new vertex to each vertex of an r-clique in G. Then $P(G';z) = (z-r)P(G;z)$.*

Proof. The identity holds when z is equal to a positive integer k, for to each proper k-colouring of G there are exactly $k-r$ colours available for the new vertex to extend to a proper colouring of G'. $\square$

Consequently, if G is a tree on n vertices then $P(G;z) = z(z-1)^{n-1}$ (every tree on $n \geq 2$ vertices has a vertex of degree 1 attached to a 1-clique in a tree on $n-1$ vertices).

A *chordal graph* is a graph such that every cycle of length four or more contains a chord, i.e., there are no induced cycles of length four or more. A chordal graph can be constructed by successively adding a new vertex and joining it to a clique of the existing graph [8]. This ordering of vertices is known as a *perfect elimination ordering.* By Proposition 7, for a chordal graph G we have $P(G;z) = z^{c(G)}(z-1)^{k_1}\cdots(z-s)^{k_s}$, where $k_1 + \cdots + k_s = |V| - c(G)$ and $s = \chi(G) - 1$.

Question 2

(i) Show that if G is the disjoint union of G_1 and G_2 then $P(G;z) = P(G_1;z)P(G_2;z)$.

(ii) Prove that

$$P(G;x+y) = \sum_{U\subseteq V} P(G[U];x)P(G[V\setminus U];y).$$

Proposition 8. *Suppose $G = (V,E)$ has the property that $V = V_1 \cup V_2$ with $G[V_1 \cap V_2]$ complete and no edges joining $V_1 \setminus (V_1 \cap V_2)$ to $V_2 \setminus (V_1 \cap V_2)$. Then*

$$P(G;z) = \frac{P(G[V_1];z)P(G[V_2];z)}{P(G[V_1 \cap V_2];z)}.$$

In particular, if G is a connected graph with 2-connected blocks $G_1,\ldots,G_\ell$ then

$$P(G;z) = z^{1-\ell}P(G_1;z)P(G_2;z)\cdots P(G_\ell;z).$$

Proof. It suffices to prove the first identity when z is a positive integer k. Each proper colouring of the clique $G[V_1 \cap V_2]$ extends to $P(G[V_1];k)/P(G[V_1 \cap V_2];k)$ proper colourings of $G([V_1])$, and independently to $P(G[V_2];k)/P(G[V_1 \cap V_2];k)$ proper colourings of $G([V_2])$. Seeing that such a proper colouring of the clique $G[V_1 \cap V_2]$ also extends to $P(G;k)/P(G[V_1 \cap V_2];k)$ proper colourings of G, we have

$$\frac{P(G;k)}{P(G[V_1 \cap V_2];k)} = \frac{P(G[V_1];k)}{P(G[V_1 \cap V_2];k)}\frac{P(G[V_2];k)}{P(G[V_1 \cap V_2];k)}.$$

$\square$

1.2 Deletion and contraction

Proposition 9. *The chromatic polynomial of a graph G satisfies the relation*

$$P(G;z) = P(G\backslash e;z) - P(G/e;z),$$

for any edge e.

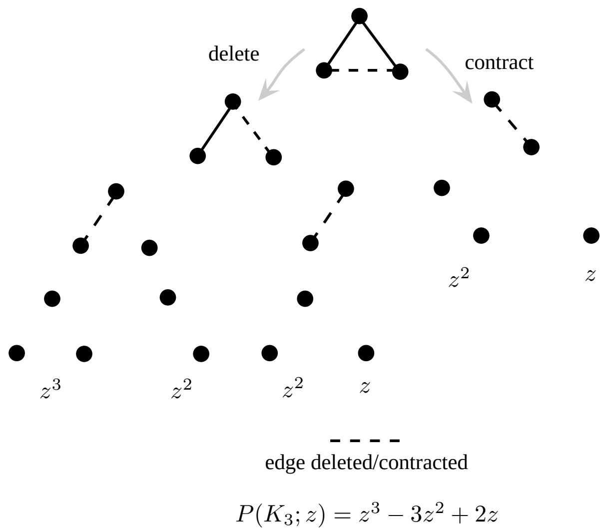


Figure 1: Deletion-contraction computation tree for the chromatic polynomial of K_3. Parallel edges produced by contraction/identification are omitted since they do not affect the value of the chromatic polynomial. Leaf nodes are empty graphs.

Proof. When e is a loop we have $P(G; z) = 0 = P(G\backslash e; z) - P(G/e; z)$ since $G\backslash e \cong G/e$. When e is parallel to another edge of G we have $P(G; z) = P(G\backslash e; z)$ and $P(G/e; z) = 0$ since G/e has a loop.

Suppose then that e is not a loop or parallel to another edge. Consider the proper vertex k-colourings of $G\backslash e$. Those which colour the ends of e differently are in bijective correspondence with proper k-colourings of G, while those that colour the ends the same are in bijective correspondence with proper k-colourings of G/e. Hence $P(G\backslash e; k) = P(G; k) + P(G/e; k)$ for each positive integer k. □

Proposition 9 provides the basis for a possible inductive proof of any given statement about the chromatic polynomial for a minor-closed class of graphs (such as planar graphs). We shall see a few such examples in this chapter.

Question 3
Use the deletion–contraction recurrence of Proposition 9 to

(i) give another proof that the chromatic polynomial of a tree on n vertices is given by $z(z-1)^{n-1}$;

(ii) find the chromatic polynomial of the cycle C_n.

We can use the recurrence given by Proposition 9 to compute the chromatic polynomial of a graph G recursively. A convenient way to record this computation is to draw a binary tree rooted at G whose nodes are minors of G and where the children of a node are the two graphs obtained by the deletion and contraction of an edge. Along each branch of the computation tree it does not matter in which order we choose the edges to delete or contract. If we continue this computation tree until no edges remain to delete and contract then the leaves of the computation tree are edgeless graphs $\overline{K}_i$ on $1 \leq i \leq n$ vertices, whose chromatic polynomial is given by z^i. The sign of this term in its contribution to the chromatic polynomial of G is positive if an even number of contractions occur on its branch, and negative otherwise. See Figure 1 for an example.

For a simple graph $G = (V, E)$ a binary deletion-contraction tree of depth $|E|$ is required to reach cocliques at all the leaves. When multiple edges appear they can be deleted to leave simple edges (in other words, contraction of an edge parallel to another edge gives a loop and this contributes zero to the chromatic polynomial).

Question 4
Suppose G is a simple connected graph on n vertices.

(i) Prove that the number of edge contractions along a branch of the computation tree for the chromatic polynomial of G whose leaf node is a coclique of i vertices is equal to $n-i$.

(ii) Prove that for each $1 \leq i \leq n$ we can always obtain a coclique on i vertices by deleting/contracting edges in some appropriate order. Deduce that

$$P(G;z) = \sum_{0\leq i\leq n-1} (-1)^i c_i(G) z^{n-i},$$

where $c_i(G) > 0$ is the number of cocliques of order $n-i$ occurring as leaf node in the computation tree for G. (A formal proof of the fact that the coefficients of $P(G;z)$ alternate in sign is given in Proposition 10 below. A combinatorial interpretation for $c_i(G)$ in terms of spanning forests of G is given by Theorem 14.)

If we start with a connected graph G in building the computation tree we can always choose an edge whose deletion leaves the graph connected, so that the children of a node are both connected graphs. In this way we end up with trees (at which point deleting any edge disconnects the tree). Seeing that we know that the chromatic polynomial of a tree on i vertices is given by $z(z-1)^{i-1}$ we could stop the computation tree at this point when we reach trees as leaf nodes. The sign of the term $z(z-1)^{i-1}$ contributed to $P(G;z)$ by a leaf node tree on i vertices is positive if there are an even number of edge contractions on its branch, and otherwise it has negative sign in its contribution. See the left-hand diagram of Figure 2 for an example with $G = K_4^-$ (K_4 minus an edge).

Question 5

(i) Show in a similar way to the previous question that if G is a connected graph on n vertices then each leaf of the deletion-contraction computation tree for G which is a tree on i vertices contributes $(-1)^{n-i}z(z-1)^{i-1}$ to $P(G;z)$.

(ii) Deduce that when G is connected

$$P(G;z) = z\sum_{1\leq i\leq n} (-1)^{n-i} t_i(G)(z-1)^{i-1},$$

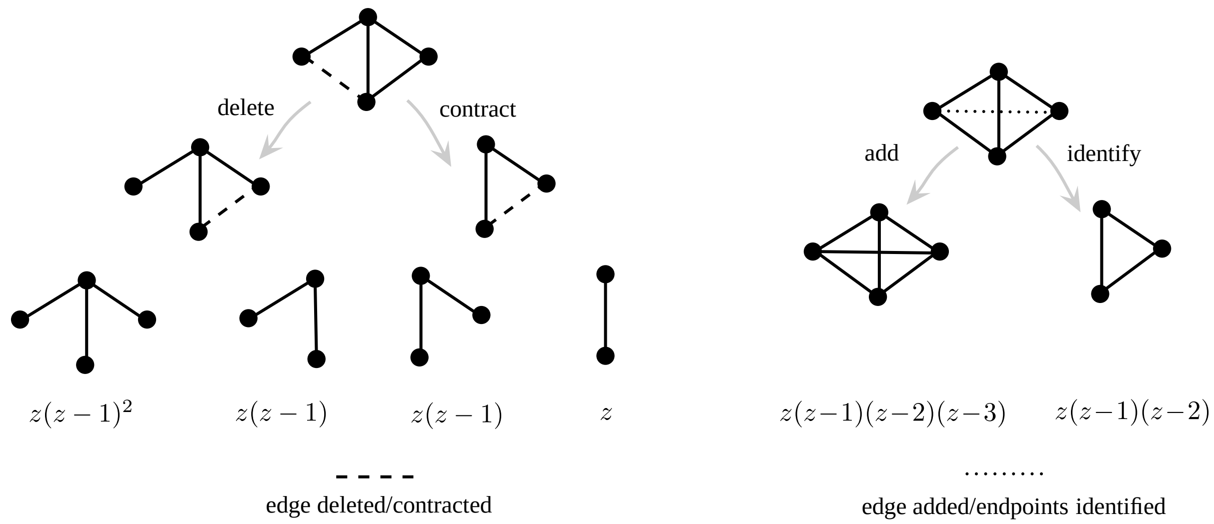
where $t_i(G)$ is the number of trees of order i occurring as leaf nodes in the computation tree for G. (We shall see in the chapter on the Tutte polynomial that the coefficients $t_i(G)$ have a combinatorial interpretation in terms of spanning trees of G.)

If we write the recurrence given in Proposition 9 as $P(G\backslash e;z) = P(G;z) + P(G/e;z)$, we can by adding edges between non-adjacent vertices or identifying such non-adjacent vertices "fill out" a dense connected graph to complete graphs. Add the edge e to $G\backslash e$ to make G, and if G/e has parallel edges these can be removed without affecting the value of $P(G/e;z)$: in any event, the number of non-edges in both G and (the simplified graph) G/e is one less than in $G\backslash e$. Hence, starting with a simple connected graph $G = (V,E)$, $\binom{|V|}{2} - |E|$ addition-identification steps are required to reach complete graphs. See the right-hand diagram in Figure 2 for a small example.

Question 6 By considering the definition of the chromatic polynomial (Definition 2), prove that

$$z^n = \sum_{1\leq i\leq n} S(n,i) z^{\underline{i}},$$

where $S(n,i)$ is equal to the number of partitions of an n-set into i non-empty sets. (These are known as the *Stirling numbers of the second kind*.)



$$P(K_4^-; z) = z(z-1)^2 - 2z(z-1) + z \qquad P(K_4^-; z) = z(z-1)(z-2)(z-3) + z(z-1)(z-2)$$

Figure 2: Deletion-contraction and addition-identification computation tree for the chromatic polynomial of K_4^-. Parallel edges produced by contraction/identification are omitted since they do not affect the value of the chromatic polynomial. Leaf nodes for deletion-contraction are trees, leaf nodes for addition-identification are complete graphs.

To move from the basis $\{z^n\}$ to the basis $\{z^{\underline{n}}\}$ for polynomials in z we have the identity

$$z^{\underline{n}} = \sum_{1 \le i \le n} s(n,i) z^i,$$

where $s(n,i)$ are the signed Stirling numbers of the first kind, defined recursively by

$$s(n,i) = s(n-1,i-1) - (n-1)s(n-1,i),$$

$$\begin{cases} s(r,0) = 0 & r = 1,2,\ldots \\ s(r,r) = 1 & r = 0,1,2,\ldots \end{cases}.$$

The number $(-1)^{n-i}s(n,i)$ counts the number of permutations of an n-set that have exactly i cycles. By Question 4 it is also the number of cocliques of order i occurring as leaves in the computation tree for $P(K_n; z)$, and by Theorem 14 below it also has an interpretation in terms of forests on n vertices.

Question 7

(i) Explain why $P(G; z) > 0$ when $z \in (-\infty, 0)$, provided G has no loops.

(ii) Show that if G is connected and without loops then $P(G; z)$ is non-zero with sign $(-1)^{|V|-1}$ when $z \in (0,1)$.

Remark Let $z^{\overline{i}}$ denote the rising factorial $z(z+1)\cdots(z+i-1)$. Brenti [5] proved that

$$P(G; z) = \sum_{1 \le i \le |V|} (-1)^{|V|-i} b_i(G) z^{\overline{i}},$$

where $b_i(G)$ is the number of set partitions $V_1 \cup V_2 \cup \cdots \cup V_i$ of V into i blocks paired with an acyclic orientation of $G[V_1] \cup G[V_2] \cup \cdots \cup G[V_i]$. See [18] for expressions for the coefficients of the chromatic polynomial relative to any polynomial basis $\{e_i(z)\}$ of binomial type (meaning it satisfies $e_j(x+y) = \sum_{0 \le i \le j} \binom{j}{i} e_i(x) e_{j-i}(y)$).

In Question 5 above you argued from the computation tree for a connected graph G that the coefficients of $P(G;z)$ alternate in sign. Let's formalize this argument and prove it for general graphs:

Proposition 10. *Suppose G is a loopless graph and that*

$$P(G;z) = \sum_{0 \le i \le |V|} (-1)^i c_i(G) z^{|V|-i}.$$

Then $c_i(G) > 0$ for $0 \le i \le r(G)$, and $c_i(G) = 0$ for $r(G) < i \le |V|$.

Proof. We shall show that

$$(-1)^{|V|} P(G;-z) = \sum_{0 \le i \le r(G)} c_i(G) z^{|V|-i}$$

has strictly positive coefficients. (When G has loops $P(G;z) = 0$.) By the deletion–contraction formula, and using the fact that $|V(G\backslash e)| = |V(G)|$ and $|V(G/e)| = |V(G)| - 1$ when e is not a loop,

$$(-1)^{|V(G)|} P(G;-z) = (-1)^{|V(G\backslash e)|} P(G\backslash e;-z) + (-1)^{|V(G/e)|} P(G/e;-z).$$

Hence

$$c_i(G) = c_i(G\backslash e) + c_{i-1}(G/e).$$

Assume inductively on the number of edges that $c_i(G) > 0$ for $0 \le i \le r(G)$, and that $c_i(G) = 0$ otherwise. As a base for induction, $(-1)^n P(\overline{K}_n; -z) = z^n$.

By inductive hypothesis, for $0 \le i \le r(G\backslash e)$ we have $c_i(G\backslash e) > 0$ and for $0 \le i-1 \le r(G/e)$ we have $c_{i-1}(G/e) > 0$. When e is not a bridge $r(G\backslash e) = r(G)$ and so $c_i(G\backslash e) > 0$ for $0 \le i \le r(G)$, otherwise for a bridge $r(G\backslash e) = r(G) - 1$ and in this case $c_i(G) > 0$ for $0 \le i \le r(G) - 1$. Since e is not a loop $r(G/e) = r(G) - 1$, so we have $c_{i-1}(G/e) > 0$ for $1 \le i \le r(G)$. Together these inequalities imply $c_i(G) > 0$ for $0 \le i \le r(G)$.

Clearly z divides $P(G;z)$ for a connected graph. It follows that $z^{c(G)}$ is a factor of $P(G;z)$ by multiplicativity of the chromatic polynomial over disjoint unions. Hence $c_i(G) = 0$ for $r(G) < i \le |V(G)|$. Also, the degree of $P(G;z)$ is $|V(G)|$ by its definition, so there are no remaining non-zero coefficients. $\square$

We shall see below in Whitney's Broken Circuit Theorem that the numbers $c_i(G)$ have a combinatorial interpretation in terms of spanning forests of G.

Question 8

(i) Prove that the only rational roots of $P(G;z)$ are $0, 1, \ldots, \chi(G) - 1$. (It may help to remind oneself that a monic polynomial with integer coefficients cannot have rational roots that are not integers.)

(ii) Show that the root 0 has multiplicity $c(G)$ and that the root 1 has multiplicity equal to the number of blocks of G.

Jackson [11] proved that $P(G;z)$ can have no root in $(1, 32/27]$. Thomassen [14] a few years later proved that in any other interval of the real line there is a graph whose chromatic polynomial has a root contained in it.

Earlier in the history of the chromatic poylnomial, Birkhoff and Lewis [3] showed that the chromatic polynomial of a plane triangulation cannot have a root in the intervals $(1,2)$ or $[5,8)$. Tutte [15] observed that for planar graphs there is often a root of the chromatic polynomial close to τ^2 where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio, and proved that if G is a triangulation of the plane with n vertices then $P(G;\tau^2) \le \tau^{5-n}$. See e.g. [9, ch. 12-14] and [12] for more about chromatic roots.

Here is another illustration of how deletion-contraction arguments can be used to give simple inductive proofs. On the other hand, as with inductive proofs generally, the art is knowing what to prove. We shall shortly see that the coefficients of $P(G;z)$ have a general expression, given by Whitney's Broken Circuit Theorem, of which Proposition 3 and the following are particular instances.

Proposition 11. *For a simple graph G on n vertices and m edges the coefficient of z^{n-2} in $P(G;z)$ is equal to $\binom{m}{2} - t$, where t is the number of triangles in G.*

Proof. The assertion is true when $m = 0, 1, 2$. Suppose G has n vertices and $m \geq 3$ edges. For a non-loop e, $c_2(G) = c_2(G\backslash e) - c_1(G/e)$. Inductively, $c_2(G\backslash e) = \binom{m-1}{2} - t_0$, where t_0 is the number of triangles in G not containing the edge e, the graph $G\backslash e$ being simple. In a triangle $\{e, e_1, e_2\}$ of G containing e, the edges e_1, e_2 do not appear in any other triangle of G containing e, since G is simple. When e is contracted the edges e_1 and e_2 become parallel edges in G/e, and moreover there are no other edge parallel to these. Hence for each triangle $\{e, e_1, e_2\}$ of G we remove one parallel edge in G/e in order to reduce it to a simple graph. So $c_1(G/e) = (m-1) - t_1$, where t_1 is the number of triangles of G containing e. With $t_0 + t_1 = t$ equal to the number of triangles in G, the result now follows by induction. □

Proposition 12. *If $P(G; z) = z(z-1)^{n-1}$ then G is a tree on n vertices, and more generally $P(G; z) = z^c(z-1)^{n-c}$ implies G is a forest on n vertices with c components.*

Proof. The degree of $P(G; z)$ is n so G has n vertices. The coefficient of z^c is non-zero but z^{c-1} has zero coefficient, hence by Proposition 10 G has c connected components. Finally, reading off the coefficient of z^{n-1} tells us that the number of edges is $n - c$, so that G is a forest on n vertices with c components. □

Question 9 Prove that if $P(G; z) = P(K_n; z)$ then $G \cong K_n$ and that if $P(G; z) = P(C_n; z)$ then $G \cong C_n$.

1.3 Subgraph expansions

From Question 21 in Chapter 4 we have by an inclusion-exclusion argument for nowhere-zero k-tensions that

$$P(G; k) = k^{c(G)} \sum_{F \subseteq E} (-1)^{|E|-|F|} k^{r(F)}.$$

A similar inclusion-exclusion argument can be used for proper vertex k-colourings:

Theorem 13. *The chromatic polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$P(G; z) = \sum_{F \subseteq E} (-1)^{|F|} z^{c(F)},$$

where $c(A)$ is the number of connected components in the spanning subgraph (V, A).

Proof. We prove the identity when z is a positive integer k.

For an edge $e = uv$ let $M_e = \{\kappa : V \to [k] : \kappa(u) = \kappa(v)\}$. Then

$$\bigcap_{e \in E} \overline{M}_e = \{\kappa : V \to [k] : \forall_{uv \in E}\ \kappa(u) \neq \kappa(v)\}$$

is the set of proper k-colourings of G. By the principle of inclusion-exclusion,

$$\left| \bigcap_{e \in E} \overline{M}_e \right| = \sum_{F \subseteq E} (-1)^{|F|} \left| \bigcap_{f \in F} M_f \right|.$$

But $\left| \bigcap_{f \in F} M_f \right| = k^{c(F)}$, since a function $\kappa : V \to [k]$ monochrome on each edge of F is constant on each connected component of (V, F), and conversely assigning each connected component a colour independently yields such a function κ. □

In the subgraph expansion for the chromatic polynomial given in Theorem 13 there are many cancellations. If $f \in F$ belongs to a cycle of (V, F) then the sets F and $F \setminus \{f\}$ have contributions to the sum that cancel. Whitney's Broken Circuit expansion results by pairing off subgraphs in a systematic way.

Let $G = (V, E)$ be a simple graph whose edge set has been ordered $e_1 < e_2 < \cdots < e_m$. A *broken circuit* is the result of removing the first edge from some circuit, i.e., a subset $B \subseteq E$ such that for some edge e_l the edges $B \cup \{e_l\}$ form a circuit of G and $i > l$ for each $e_i \in B$.

Theorem 14. Whitney [17]. *Let G be a simple graph on n vertices with edges totally ordered, and let $P(G;z) = \sum(-1)^i c_i(G)z^{n-i}$. Then $c_i(G)$ is the number of subgraphs of G which have i edges and contain no broken circuits.*

Proof. Suppose $B_1, \ldots, B_t$ is a list of the broken circuits in lexicographic order based on the ordering of E. Let f_j $(1 \leq j \leq t)$ denote the edge which when added to B_j completes a circuit. Note that $f_j \notin B_k$ when $k \geq j$ (otherwise B_k would contain in f_j an edge smaller than any edge in B_j, contrary to lexicographic ordering).

Define $\mathcal{S}_0$ to be the set of subgraphs of G containing no broken circuit and for $1 \leq j \leq t$ define $\mathcal{S}_j$ to be the set of subgraphs containing B_j but not B_k for $k > j$. Then $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_t$ is a partition of the set of all subgraphs of G.

If $A \subseteq E$ does not contain f_j, then A contains B_j if and only if $A \cup \{f_j\}$ contains B_j. Further, A contains B_k $(k > j)$ if and only if $A \cup \{f_j\}$ contains B_k, since f_j is not in B_k either. If one the subgraphs A and $A \cup \{f_j\}$ are in $\mathcal{S}_j$ then both are, and since $c(A) = c(A \cup \{f_j\})$ the contributions to the alternating sum cancel.

The only terms remaining are contributions from subsets in $\mathcal{S}_0$: a subset of size i spans a forest with $n - i$ components, thus contributing $(-1)^i z^{n-i}$ to the sum. □

Proposition 15. *Suppose G is a simple connected graph on n vertices and m edges and having girth g, and that $P(G;z) = \sum(-1)^i c_i(G)z^{n-i}$. Then*

$$c_i(G) = \binom{m}{i}, \text{ for } i = 0, 1, \ldots, g-2,$$

and

$$c_{g-1}(G) = \binom{m}{g-1} - t,$$

where t is the number of circuits of size g in G.

Question 10
Show that if G is a simple connected graph on n vertices and m edges and $P(G;z) = \sum(-1)^i c_i(G)z^{n-i}$ then, for $0 \leq i \leq n-1$,

$$\binom{n-1}{i} \leq c_i(G) \leq \binom{m}{i}.$$

Proposition 16. *If G is a simple connected graph on n vertices and m edges and $P(G;z) = \sum(-1)^i c_i(G)z^{n-i}$ then,*

$$c_{i-1}(G) \leq c_i(G) \ \textit{ for all } \ 1 \leq i \leq \frac{1}{2}(n-1).$$

Proof. In terms of the coefficients relative to the tree basis $\{z(z-1)^{n-1}\}$,

$$P(G;z) = \sum_{i=1}^{n} (-1)^{n-i} t_i(G) z(z-1)^{i-1},$$

we have

$$c_i(G) = \sum_{0 \leq j \leq i} t_{n-j}(G) \binom{n-1-j}{n-1-i} = \sum_{j=0}^{i} t_{n-j}(G) \binom{n-1-j}{i-j}.$$

If $i \leq \frac{1}{2}(n-1)$ then $i - j \leq \frac{1}{2}(n-1-j)$ for all $j \geq 0$. By unimodality of the binomial coefficients,

$$\binom{n-1-j}{i-j} \geq \binom{n-1-j}{i-1-j} \text{ for } i \leq \frac{1}{2}(n-1),\ j \geq 0.$$

Since each $t_{n-j}(G)$ is a non-negative integer, it follows that $c_i(G) \geq c_{i-1}(G)$ for $i \leq \frac{1}{2}(n-1)$. □

Question 11

Recall that if G is a forest then $P(G;z) = z^{c(G)}(z-1)^{r(G)}$. Also $(-1)^{|V(G)|}P(G;-z) = \sum_i c_i(G)z^{|V(G)|-i}$, where $c_i(G) = c_i(G\backslash e) + c_{i-1}(G/e)$.

(i) Simplify the proof of Proposition 10, that $c_i(G) > 0$ for $0 \leq i \leq r(G)$ and $c_i(G) = 0$ otherwise, by using as base for induction the truth of the statement for forests and choosing a non-bridge edge in the deletion-contraction induction step.

(ii) Likewise, prove that $c_{i-1}(G) < c_i(G)$ for $0 \leq i \leq \frac{1}{2}r(G)$ (Proposition 16 for not necessarily connected graphs G) by using base for induction the fact that this statement is true for forests and using deletion-contraction of a non-bridge edge.

(iii) Re-prove Theorem 14 that $c_i(G)$ is the number of i-subsets of $E(G)$ not containing a broken circuit by showing that this quantity satisfies the recurrence $c_i(G) = c_i(G\backslash e) + c_{i-1}(G/e)$. (For this induction on number of edges the base case is $c_0(\overline{K}_n) = 1$ and $c_i(\overline{K}_n) = 0$ for $i > 0$, for which the assertion is trivially satisfied. To move by induction to an arbitrary graph G, with total order on $E(G)$ used to define broken circuits, choose the edge e to be the greatest.)

Proposition 16 is the easy half of a long-standing conjecture first made by Read in 1968 that the coefficients $c_i(G)$ of the chromatic polynomial are *unimodal.* An even stronger conjecture of *log-concavity* was later made, i.e., that $c_{i-1}(G)c_{i+1}(G) \leq c_i(G)^2$. Both conjectures fell simultaneously in 2010 when J. Huh [10] proved log-concavity as a corollary of a more general theorem in algebraic geometry.

A theorem due to Newton states that if a polynomial $\sum_i c_i z^{n-i}$ has strictly positive coefficients and all of its roots are real then the sequence (c_i) of coefficients is log-concave (and hence unimodal). If it were the case that the chromatic polynomial always had real roots then log-concavity of the sequence of absolute values of its coefficients would therefore follow by this result. However, not only is it true that there are some graphs whose chromatic polynomial has complex roots that are not real, but Sokal [13] showed that the set of complex numbers that are roots of some chromatic polynomial are dense in the whole complex plane. (This in contradistinction to when we restrict attention to the real line itself, where no chromatic roots can lie on $(-\infty, \frac{32}{27}]$.) Can you think of a family of graphs $\{G_n\}$ with the property that $P(G_n;z)$ has non-real roots?

1.4 Some other deletion–contraction invariants.

We have seen that the chromatic polynomial $P(G;z)$ satisfies the recurrence relation

$$P(G;z) = P(G\backslash e;z) - P(G/e;z), \tag{1}$$

for any edge e of G. Together with boundary conditions

$$P(\overline{K}_n;z) = z^n, \qquad n = 1, 2, \ldots \tag{2}$$

this suffices to determine $P(G;z)$ on all graphs. A slight variation on giving the boundary conditions (2) is to supplement the recurrence (1) with the property of multiplicativity over disjoint unions

$$P(G_1 \cup G_2;z) = P(G_1;z)P(G_2;z), \tag{3}$$

and then to give the single boundary condition $P(K_1;z) = z$.

Define

$$B(G;k,y) = \sum_{f:V(G)\to[k]} y^{\#\{uv\in E(G):f(u)=f(v)\}},$$

where $k \in \mathbb{Z}_{>0}$ and y is an indeterminate. This polynomial in y is a generating function for colourings of G (not necessarily proper) counted according to the number of monochromatic edges, i.e., edges receiving the same colour on their endpoints. (Edges are taken with their multiplicity when counting the number of monochromatic edges in the exponent of y.) Note that $B(G;k,0) = P(G;k)$.

Proposition 17. *For each edge e of G,*

$$B(G;k,y) = (y-1)B(G/e;k,y) + B(G\backslash e;k,y).$$

Together with the boundary conditions $B(\overline{K}_n;k,y) = k^n$, for $n = 1,2,\ldots,$ this determines $B(G;k,y)$ as a polynomial in k and y.

Proof. Given $e = st$,

$$\begin{aligned} B(G;k,y) &= y \sum_{\substack{f:V(G)\to[k] \\ f(s)=f(t)}} y^{\#\{uv\in E\backslash e: f(u)=f(v)\}} + \sum_{\substack{f:V(G)\to[k] \\ f(s)\neq f(t)}} y^{\#\{uv\in E\backslash e: f(u)=f(v)\}} \\ &= yB(G/e;k,y) + [B(G\backslash e;k,y) - B(G/e;k,y)]. \end{aligned}$$

The fact that $B(G;k,y)$ is a polynomial follows by induction of the number of edges and the given boundary condition $B(\overline{K}_n;k,y) = k^n$. Further, it has degree $|V(G)|$ as a polynomial in k and degree $|E(G)|$ as a polynomial in y (again by induction on number of edges by tracking the relevant coefficient in the recurrence $B(G;k,y) = (y-1)B(G/e;k,y) + B(G\backslash e;k,y)$). □

An *acyclic orientation* of a graph is an orientation that has no directed cycles. A loop has no acyclic orientation, but any loopless graph does (for example, if its vertices are labelled by $1,\ldots,n$ and an edge is directed from the smaller to the higher number).

Theorem 18. [Stanley, 1973] *The number of acyclic orientations of a graph G with at least one edge is given by $(-1)^{|V(G)|}P(G;-1)$.*

Proof. Let $Q(G)$ denote the number of acyclic orientations of G. When G is a single edge $Q(G) = 2$ and when G is a loop $Q(G) = 0$. If e is parallel to another edge of G then $Q(G) = Q(G\backslash e)$, since parallel edges must have the same direction in an acyclic orientation. Also, Q is multiplicative over disjoint unions, i.e., $Q(G_1 \cup G_2) = Q(G_1)Q(G_2)$.

To prove then that $Q(G) = (-1)^{|V(G)|}P(G;-1)$ it suffices to show that when e is not a loop or parallel to another edge of G we have

$$Q(G) = Q(G\backslash e) + Q(G/e). \tag{4}$$

Let $e = uv$ be a simple edge of G and consider an acyclic orientation $\mathcal{O}$ of $G\backslash e$. There is always one direction $u \to v$ or $u \leftarrow v$ possible so that $\mathcal{O}$ can be extended to an acyclic orientation of G: if both directions were to produce directed cycles then there would have to be a directed path from u to v and a directed path from v to u, which together would make a directed cycle in $\mathcal{O}$.

Those acyclic orientations of $G\backslash e$ that permit exactly one direction of e are in bijective correspondence with the subset of acyclic orientations of G where the direction of e cannot be reversed while preserving the property of being acyclic. Such an orientation of G induces an orientation that has a directed cycle in G/e, and contributes 1 to $Q(G)$ and $1+0=1$ to $Q(G\backslash e) + Q(G/e)$.

Those acyclic orientations of $G\backslash e$ where the direction of e can be reversed to make another acyclic orientation of G are in bijective correspondence with those orientations of G that induce acyclic orientations on the contracted graph G/e. Such a pair of acyclic orientations of G differing just on the direction of e contribute 2 to $Q(G)$ and $1+1=2$ to $Q(G\backslash e) + Q(G/e)$.

This establishes the recurrence (4). □

In [16] Tutte describes how he was led to define his polynomial (he called it the dichromate) by observing how graph invariants such as the chromatic polynomial and the number of spanning trees of a graph shared the property of satisfying a deletion–contraction recurrence.

Question 12 Suppose $f(G)$ is a graph invariant that for a connected graph G counts one of the following:

(i) the number of spanning trees of G,

(ii) the number of spanning forests of G,

(iii) the number of connected spanning subgraphs of G.

Further suppose we stipulate that f is multiplicative over disjoint unions, $f(G_1 \cup G_2) = f(G_1)f(G_2)$.
Show that in each case f satisfies the recurrence

$$f(G) = f(G \backslash e) + f(G/e),$$

for each edge e of G that is not a loop or bridge. How do these three invariants differ for bridges and loops?

These deletion-contraction invariants form a sort of preview of the chapter on the Tutte polynomial. Before that though we return to the flow polyomial and see how some of the properties of the chromatic polynomial dualize to properties of the flow polynomial.

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