Combinatorics and Graph Theory I

Exercise sheet 6: Graph connectivity

12 April 2017

1. Let $\delta(G)$ denote the minimum degree of graph G.

(i) Define the parameters $\kappa(G)$ and $\lambda(G)$.

(ii) Prove that

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

for a graph G on more than one vertex.

[Bollobás, *Modern Graph Theory*, III.2.]

(iii) Let k and ℓ be integers with $1 \leq k \leq \ell$.

(a) Construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

(b) Construct a graph G with $\kappa(G) = k$ and $\kappa(G - v) = \ell$ for some vertex v.

(a) Construct a graph G with $\lambda(G - u) = k$ and $\lambda(G - uv) = \ell$ for some edge uv.

[Bollobás, *Modern Graph Theory*, III.6, exercise 11]

2. Given $U \subset V(G)$ and a vertex $x \in V(G) - U$, an $x - U$ *fan* is a set of $|U|$ paths from x to U any two of which have exactly the vertex x in common. Prove that a graph G is k-connected iff $|G| \geq k + 1$ and for any $U \subset V(G)$ of size $|U| = k$ and vertex x not in U there is an $x - U$ fan in G.

[*Given a pair (x, U), add a vertex u to G and join it to each vertex in U. Check that the new graph is k-connected if G is. Apply Menger's theorem for x and u.*]

[Bollobás, *Modern Graph Theory*, III.6 exercise 13]

3. Prove that if G is k-connected ($k \geq 2$), then every set of k vertices is contained in a cycle. Is the converse true?

[Bollobás, *Modern Graph Theory*, III.6 exercise 14. Cf. for $k = 2$, Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd, ed. Theorem 4.6.3]

4. The line graph $L(G)$ of a graph $G = (V, E)$ has vertex set E and two vertices $e, f \in E$ are adjacent iff they have exactly one vertex of G in common. By applying the vertex form of Menger's theorem to the line graph $L(G)$, prove that the vertex form of Menger's theorem implies the edge form.

[Bollobás, *Modern Graph Theory*, III.6 exercise 15.]