Combinatorics and Graph Theory I

Exercise sheet 5: Bipartite matching

5 April 2017

1. An *augmenting path* with respect to a matching M in a bipartite graph G is a path P in G which starts at an unmatched vertex and then contains alternately an edge not in M and an edge in M and ends in another unmatched vertex. The symmetric difference of the edges of P and the edges of M is again a matching and covers two more vertices than M (the endpoints of P).

(i) Let M be a matching in a bipartite graph G. Show that if M contains fewer edges than some other matching N in G then G contains an augmenting path with respect to M.

[Consider the symmetric difference of M and N.]

(ii) Describe an algorithm based on (i) that finds a matching of maximum cardinality in any given biparite graph. [*Fine details not required; you may assume there is an oracle that provides you with an augmenting path when one exists.*]

[Diestel, *Graph Theory* 2nd ed., 2.1, chapter 2 exercises 1, 2]

2.

(a) Use Hall's condition to show that the bipartite graph in Figure 1 has no complete matching.

(b) Let M be the matching $\{x_3y_2, x_4y_4, x_5y_5\}$ denoted by heavier lines in Figure 1.

(i) Find an alternating path for M beginning at x_2.

(ii) Use it to construct a matching M' with four edges.

(iii) Check that there is no alternating path for M'.

(iv) Is M' a maximum matching? (i.e. are there any matchings with more than four edges?)

[Biggs, *Discrete Mathematics*, exercises 10.4.1 and 10.4.2]

3. Let G be a bipartite graph with vertex sets V_1 and V_2. Let U be the set of vertices of maximal degree (i.e., the degree of each vertex in U is the maximum degree of G). Show that there is a complete matching from $U \cap V_1$ into V_2.

[Bollobás, *Modern Graph Theory*, III.6 exercise 21.]

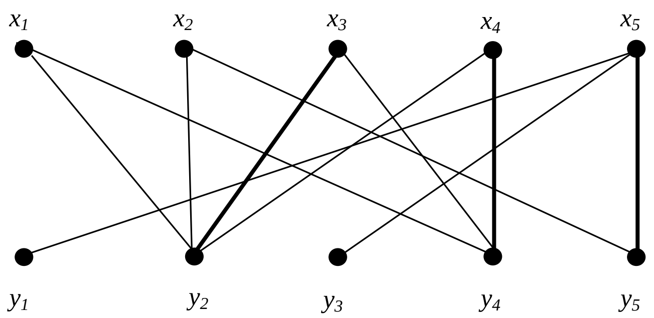

Figure 1: Bipartite graph for Exercise 2.

4. Let $G = (V, E)$ be a bipartite graph with vertex classes X and Y of sizes m and n that contains a complete matching from X to Y.

(i) Prove that there is a vertex $x \in X$ such that for every edge xy there is a matching from X to Y that contains xy.

(ii) Deduce that if $d(x) = d$ for every $x \in X$ then G contains at least $d!$ complete matchings if $d \leq m$ and at least $d(d-1)\cdots(d-m+1)$ complete matchings if $d > m$.

[Bollobás, *Modern Graph Theory*, III.6 exercise 18.]

5. Let $A = (a_{i,j})$ be an $n \times n$ *doubly stochastic matrix*, i.e., $a_{i,j} \geq 0$ for all i, j and

$$\sum_{i=1}^{n} a_{i,j} = 1 = \sum_{j=1}^{n} a_{i,j}.$$

A special case of a doubly stochastic matrix is a permutation matrix P in which all but one entry in each row is equal to 0, the other being 1 (and hence the same is true of each column).

(a) Let $a^*_{i,j} = \lceil a_{i,j} \rceil$ (equal to 1 if $a_{i,j} \neq 0$ and 0 if $a_{i,j} = 0$) and $A^* = (a^*_{i,j})$ be the bipartite adacency matrix of a bipartite graph G with vertex classes both of size n (thus, ij is an edge iff $a^*_{i,j} = 1$).

Show that G has a complete matching.

(b) Deduce from (a) that there are a permutation matrix P and a real λ, $0 < \lambda \leq 1$, such that $A - \lambda P = B = (b_{i,j})$ satisfies $b_{i,j} \geq 0$ for all i, j and

$$\sum_{i=1}^{n} b_{i,j} = 1 - \lambda = \sum_{j=1}^{n} b_{i,j}.$$

Further, explain why B has at least one more 0 entry than A.

(c) Deduce from (b) that there are $\lambda_i \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ and permutation matrices $P_1, P_2, \ldots, P_m$ such that

$$A = \sum_{i=1}^{m} \lambda_i P_i.$$

(This is to say that A lies in the *convex hull* of the $n \times n$ permutation matrices.)

[Bollobás, *Modern Graph Theory*, III.6 exercise 19.]