Combinatorics and Graph Theory I

Exercise sheet 3: Generating functions ctd.

8 March 2017

1. Express the nth term of the sequences given by the following recurrence relations (generalize the method used for the Fibonacci numbers in Section 12.3):

(a) $a_0 = 2$, $a_1 = 3$, $a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \dots)$

(b) $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 4a_{n+1} - 4a_n$ $(n = 0, 1, 2, \dots)$

(c) $a_0 = 1$, $a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \dots)$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.3.]

2. Solve the recurrence $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$. Find $\lim_{n\to\infty} a_n$.

[*Take base 2 logarithms of the given recurrence.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.5.]

3.

(a) Solve the recurrence $a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$ with the initial condition $a_0 = 1$.

*(b) Solve the recurrence $a_n = a_{n-1} + \ a_{n-3} + \cdots + a_1 + a_0$ $(n \geq 3)$ with the initial condition $a_0 = a_1 = a_2 = 1$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.6.]

4. Express the sum

$$S_n = \binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n}$$

as the coefficient of x^{2n} in a suitable power series. Find a simple formula for S_n.

[*Use* $x^k(1+2x)^k = \sum_{i=0}^{k} 2^i\binom{k}{i}x^{k+i}$*, sum over integers* k*, and set* $k+i = 2n$ *to pick out the requisite coefficient.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.7]

5.

(a) Show that the number $\frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n$ is an integer for all $n \geq 1$. [*Find the generating function for the sequence; equivalently, the recurrence it satisfies.*]

(b) Show that the decimal expansion of $(6+\sqrt{37})^{999}$ has at least 999 zeros following the decimal point.

[*Show that* $a_n = (6+\sqrt{37})^n + (6-\sqrt{37})^n$ *satisfies* $a_{n+2} = 12a_{n+1} + a_n$ *with initial conditions* $a_0 = 2$, $a_1 = 12$*, and hence is an integer for all* $n \geq 1$*. Use the fact that* $\sqrt{37} - 6 < 0 \cdot 1$.]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercises 12.3.9 and 12.3.10.]