Combinatorics and Graph Theory I

Exercise sheet 8: Latin squares, Pigeonhole-Principle, Ramsey theory

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1. Read Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., section 9.3 (Orthogonal Latin Squares).

(i) Prove that the $n \times n$ array L whose (i, j)-entry is defined by

$$L(i, j) = i + j \pmod n$$

is a Latin square.

(ii) Let p be a prime and $1 \leq k \leq p - 1$. Prove that the $p \times p$ array L_k whose (i, j)-entry is defined by

$$L_k(i, j) = ki + j \pmod p$$

defines a Latin square.

(iii) Prove that when $k \neq \ell$ the Latin squares L_k and L_ℓ defined in (ii) are orthogonal.

[Adaptation of Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 9.3, exercise 5, which uses the same construction for a finite field on a prime power number of elements more generally; here the finite field is $\mathbb{Z}_p$.]

2. Use the Pigeonhole Principle to show that any finite graph has at least two vertices of the same degree.

[P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms,* Cambridge Univ. Press, 1994. Chapter 10, exercise 2]

3. The Erdős–Szekeres Theorem states that given $n \geq (r - 1)(s - 1) + 1$ distinct real numbers $x_1, x_2, ..., x_n$ there is either a strictly increasing subsequence of length r, or a strictly decreasing subsequence of length s.

(i) Show that if $n \geq (r - 1)(s - 1)(t - 1) + 1$ then any sequence of n real numbers (not necessarily distinct) must contain either a strictly increasing subsequence of length r, a strictly decreasing subsequence of length s, or a constant subsequence of length t.

[*First consider the case where only* $(r - 1)(s - 1)$ *or fewer distinct values occur and apply the Pigeonhole Principle to deduce the existence of a suitably long constant subsequence. Otherwise there are at least* $(r - 1)(s - 1) + 1$ *distinct elements....*]

(ii) Show also that the result of (i) is best possible, i.e., construct a sequence of $(r - 1)(s - 1)(t - 1)$ real numbers with no strictly increasing subsequence of length r, no strictly decreasing subsequence of length s, and no constant subsequence of length t.

[P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms,* Cambridge Univ. Press, 1994. Chapter 10, exercise 4]

4. For $n \in \mathbb{N}$ define

$$f(n) = \min_{G:|V(G)|=n} [\alpha(G)\omega(G)],$$

where the minimum is over all graphs G with n vertices, $\omega(G)$ is the largest number of mutually adjacent vertices in G (clique number), and $\alpha(G)$ is the largest number of mutually non-adjacent vertices in G (independence number). So for example $f(2) = \min\{2 \cdot 1, 1 \cdot 2\} = 2$ (G is either a single edge K_2 or its complement).

(i) Show that for $n \in \{1, 2, 3, 4, 6\}$ we have $f(n) = n$.

(ii) Prove that $f(5) < 5$.

(iii) Show that $f(n)$ is nondecreasing and that it is not bounded above by a constant.

(iv)* For natural numbers n, k, $1 \leq k < n/2$ we define a graph $C_{n,k}$ as follows. We begin with C_n, i.e., a cycle of length n, and then we connect by edges all pairs of vertices that have distance at most k in C_n (thus $C_{n,1} = C_n$). Use these graphs (with a judicious choice of k) to prove that $f(n) < n$ for all $n \geq 7$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 11.2, exercises 2, 3]