

Combinatorics and Graph Theory I

Exercise sheet 6: Spanning trees and double counting

9 April 2018

1. Let $\tau(G)$ denote the number of spanning trees of a connected graph G . Cayley's formula states that $\tau(K_n) = n^{n-2}$ for $n \geq 2$. Let K_n^- denote a graph isomorphic to K_n with one edge removed.

Find a formula for $\tau(K_n^-)$.

[*Hint*: the number of spanning trees containing a given edge of K_n is by symmetry the same for all edges.]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 8.1, exercise 2]

2.

(i) Determine natural numbers a and b with $a + b = n$ for which the product ab is maximized.

(ii) For natural numbers k and n , determine all values of natural numbers a_1, \dots, a_k satisfying $\sum_{i=1}^k a_i = n$ such that the product $a_1 a_2 \cdots a_k$ is maximized.

(iii) A complete k -partite graph $K(V_1, V_2, \dots, V_k)$ on a vertex set V is determined by a partition V_1, \dots, V_k of the set V , in which edges are pairs $\{x, y\}$ of vertices such that x and y lie in different classes of the partition. Formally, $K(V_1, \dots, V_k) = (V, E)$, where $\{x, y\} \in E$ exactly if $x \neq y$ and $|\{x, y\} \cap V_i| \leq 1$ for all $i = 1, \dots, k$. Using part (ii), prove that the maximum number of edges of a complete k -partite graph on a given vertex set corresponds to a partition with almost equal parts, i.e. one with $||V_i| - |V_j|| \leq 1$ for all i, j . How many edges are there in such a graph $K(V_1, \dots, V_k)$?

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 4.7, exercises 1,2 and 3]

3. Prove that for any $t \geq 2$, the maximum number of edges of a graph on n vertices containing no $K_{2,t}$ as a subgraph is at most

$$\frac{1}{2} \left(\sqrt{t-1} n^{3/2} + n \right).$$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 7.3, exercise 1]