# Combinatorics and Graph Theory I

## Exercise sheet 5: Connectivity, ear decompositions

26 March 2018 (handing in work for this sheet is optional)

**Reference**: Chapter III (Flows, Connectivity and Matching), Section 2 (Connectivity and Menger's Theorem) in Bollobás *Modern Graph Theory*, 1998. Section 4.6 (2-connectivity) in Matoušek and Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 2009.

1. The line graph $L(G)$ of a graph $G = (V, E)$ has vertex set $E$ and two vertices $e, f \in E$ are adjacent iff they have exactly one vertex of $G$ in common. By applying the vertex form of Menger's theorem to the line graph $L(G)$, prove that the vertex form of Menger's theorem implies the edge form.

[Bollobás, *Modern Graph Theory*, III.6 exercise 15.]

2. Let $G = (V, E)$ be a bipartite graph with vertex classes $X$ and $Y$ of sizes $m$ and $n$ that contains a complete matching from $X$ to $Y$.

(i) Prove that there is a vertex $x \in X$ such that for every edge $xy$ there is a matching from $X$ to $Y$ that contains $xy$.

(ii) Deduce that if $d(x) = d$ for every $x \in X$ then $G$ contains at least $d!$ complete matchings if $d \leq m$ and at least $d(d-1)\cdots(d-m+1)$ complete matchings if $d > m$.

[Bollobás, *Modern Graph Theory*, III.6 exercise 18.]

3. Prove that if $G$ is $k$-connected ($k \geq 2$), then every set of $k$ vertices is contained in a cycle. Is the converse true?

[Bollobás, *Modern Graph Theory*, III.6 exercise 14. Cf. for $k = 2$, Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd, ed. Theorem 4.6.3]

4. Let $G$ be a *critical 2-connected graph*, i.e. $G$ is 2-connected but no graph $G - e$ for $e \in E(G)$ is 2-connected.

(a) Prove that at least one vertex of $G$ has degree 2.

(b) For each $n$, find an example of a critical 2-connected graph with a vertex of degree at least $n$.

(c)* For each $n$, give an example of a critical 2-connected graph with a vertex of degree $\geq n$ that is at distance at least $n$ from each vertex of degree 2.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. Section 4.6, Exercise 2]

5. Answer, with justification, the following two questions concerning ear decompositions:

   (a) Is it true that any critical 2-connected graph can be obtained from a cycle by successive gluing of "ears" of length at least 2?

   (b) Is it true that any critical 2-connected graph can be obtained from a cycle by successive gluing of "ears" in such a way that each of the intermeidate graphs created along the way is also 2-critical?

   [Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. Section 4.6, Exercise 3]