Combinatorics and Graph Theory I

Exercise sheet 2: Generating functions

5 March 2018 (to hand in 12 March)

Reference: Chapter 12 (Generating Functions) in the 2nd edition of Matoušek & Nešetřil, *Invitation to Discrete Mathematics* (Chapter 10 in the 1st ed.)

1. For a polynomial or power series $a(x)$, $[x^k]a(x)$ denotes the ceofficient of x^k in $a(x)$. Determine the following coefficients:

(a) $[x^5](1-2x)^{-2}$

(b) $[x^4]\sqrt[3]{(1+x)}$

c) $[x^3]\,(2+x)^{3/2}/(1-x)$

(d) $[x^3](1-x+2x^2)^9$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.2]

2. Find generating functions for the following sequences (express them in closed form, without infinite series!):

(a) $0, 0, 0, 0, -6, 6, -6, 6, -6, \ldots$

(b) $1, 0, 1, 0, 1, 0, \ldots$

(c) $1, 2, 1, 4, 1, 8, \ldots$

(d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.3]

3.

(a) Let $A, B, C \subset \mathbb{N}$ and let $a(x) = \sum_{i\in A} x^i$, $b(x) = \sum_{j\in B} x^j$ and $c(x) = \sum_{k\in C} x^k$ be power series. Explain why the number of solutions to the equation $i + j + k = n$ with $i \in A$, $j \in B$ and $k \in C$ is equal to the coefficient of x^n in the power series $a(x)b(x)c(x)$.

(b) Let a_n be the number of solutions to the equation

$$i + 3j + 3k = n, \qquad i \geq 0, j \geq 1, k \geq 1.$$

Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and derive a formula for a_n.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.5]

4.

(a) Find the generating function for the sequence $(a_0, a_1, a_2, \dots)$ with $a_n = (n+1)^2$.

(b) Check that if $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \dots)$ then $\frac{1}{1-x}a(x)$ is the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots)$.

(c) Using (a) and (b) calculate the sum $\sum_{k=1}^{n} k^2$.

(d) By a similar method, calculate the sum $\sum_{k=1}^{n} k^3$.

(e) For natural numbers n and m, find a closed formula for the sum $\sum_{k=0}^{m} (-1)^k \binom{n}{k}$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, Problem 2.2.4, exercise 12.2.9.]

5. Express the nth term of the sequences given by the following recurrence relations (generalize the method used for the Fibonacci numbers in Section 12.3):

(a) $a_0 = 2$, $a_1 = 3$, $a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \dots)$

(b) $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 4a_{n+1} - 4a_n$ $(n = 0, 1, 2, \dots)$

(c) $a_0 = 1$, $a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \dots)$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.3.]

6. Express the sum

$$S_n = \binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n}$$

as the coefficient of x^{2n} in a suitable power series. Find a simple formula for S_n.

[*Use* $x^k(1+2x)^k = \sum_{i=0}^{k} 2^i \binom{k}{i} x^{k+i}$, *sum over integers* k, *and set* $k + i = 2n$ *to pick out the requisite coefficient.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.7]