A Tutte polynomial for maps

Andrew Goodall[1]
Thomas Krajewski Bart Litjens Guus Regts Lluís Vena

[1]Charles University, Prague

27th Workshop *Cycles and Colourings*, High Tatras, Slovakia
6 September 2018, Nový Smokovec

Seventy years ago...

Tutte 1947, 1954.

- G-flows of a graph (taking values in abelian group G).
  Nowhere-zero G-flows counted by flow polynomial.
- G-tensions $\leftrightarrow$ colourings.
  Proper colourings/Nowhere-zero G-tensions counted by chromatic polynomial.
- Unified in the bivariate dichromate (Tutte polynomial):
  includes chromatic polynomial and flow polynomial as specializations

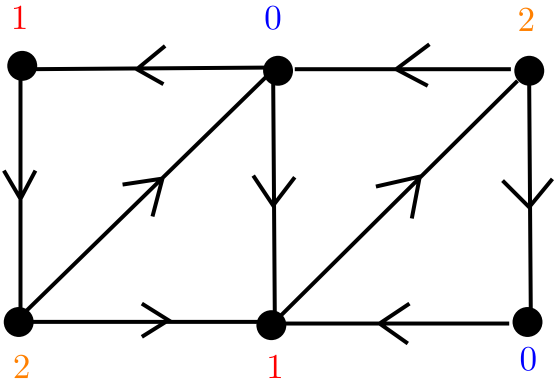
Today...

- Flows and and tensions for maps (graphs 2-cell embedded in closed surfaces) taking values in finite group G.
- G-flows and G-tensions counted by multivariate polynomial in $|G|/d_\rho$ for d_ρ dimensions of irreducible representations ρ of G
- Surface Tutte polynomial for maps.
- Evaluations give number of nowhere-identity G-flows and nowhere-identity G-tensions of a map

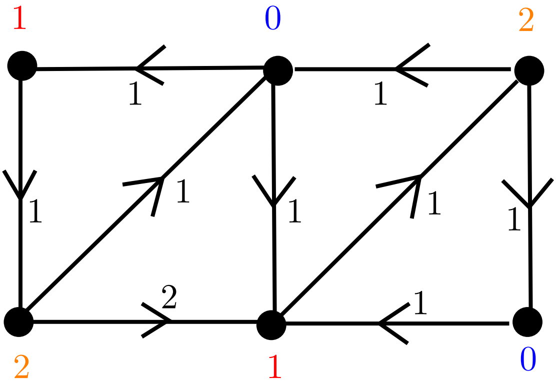
Today...

- Flows and and tensions for maps (graphs 2-cell embedded in closed surfaces) taking values in finite group G.
- G-flows and G-tensions counted by multivariate polynomial in $|G|/d_\rho$ for d_ρ dimensions of irreducible representations ρ of G
- Surface Tutte polynomial for maps.
- Evaluations give number of nowhere-identity G-flows and nowhere-identity G-tensions of a map
- Specializes to Tutte polynomial of underlying graph.
- Specializes to a new Tutte polynomial for signed graphs, which includes as evaluations the number of nowhere-zero flows, nowhere-zero tensions, and signed graph colourings.

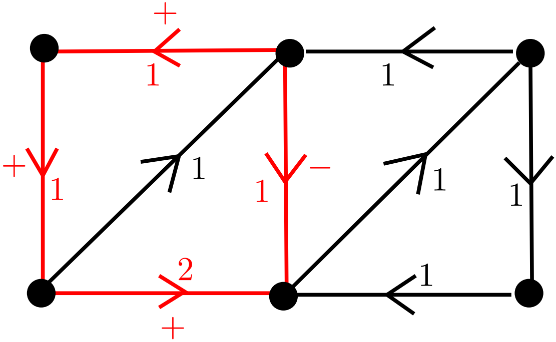
Proper vertex 3-colouring



Nowhere-zero $\mathbb{Z}_3$-tension



Nowhere-zero $\mathbb{Z}_3$-tension



$1 + 1 + 2 - 1 = 0$ in $\mathbb{Z}_3$

Colourings and tensions of graphs

Graph $\Gamma = (V, E)$ with fixed orientation of edges.
Abelian group G (written additively).

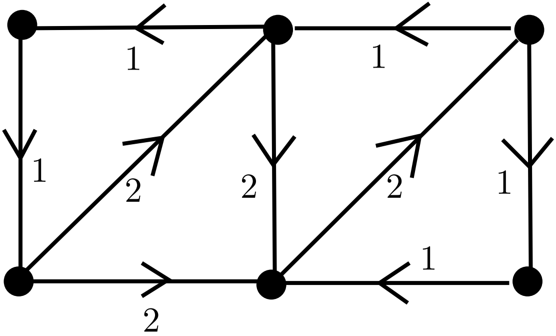
- Colour vertices of Γ with elements of G.
- Use orientation to define tension values on E.
- Defining property of tensions: net value around any directed closed walk is zero (does not refer to vertex colouring).
- Suffices to check irreducible closed walks: edges in a circuit (of cycle matroid of Γ)
- Nowhere-zero tension corresponds to proper colouring.

Counting colourings and tensions

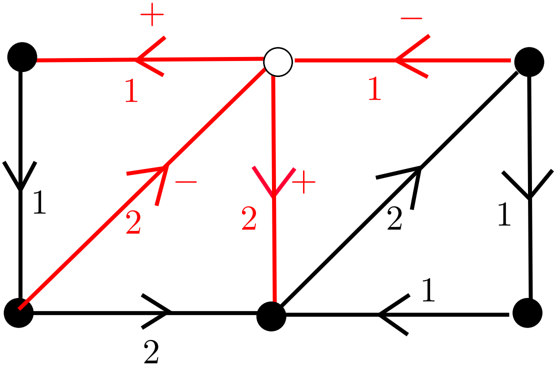
Graph Γ with $k(\Gamma)$ connected components.
deletion $\Gamma\backslash e$ and contraction Γ/e

- number of proper G-colourings is $\chi(\Gamma;|G|)$ (chromatic polynomial evaluated a $|G|$)
- $\chi(\Gamma) = \begin{cases} \chi(\Gamma\backslash e) - \chi(\Gamma/e) & \text{for } e \text{ not a loop} \\ 0 & \text{for } e \text{ a loop.} \end{cases}$
- $|G|^{-k(\Gamma)}\chi(\Gamma;|G|) = \#\{\text{nowhere-zero } G\text{-tensions of } \Gamma\}$ for any finite abelian group G

Nowhere-zero $\mathbb{Z}_3$-flow

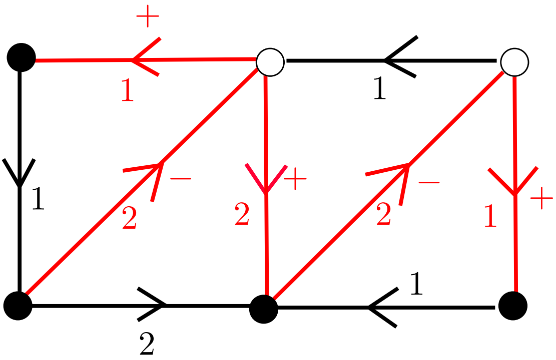


Nowhere-zero $\mathbb{Z}_3$-flow



$1 + 2 - 1 - 2 = 0$ in $\mathbb{Z}_3$

Nowhere-zero $\mathbb{Z}_3$-flow



$1 + 2 + 1 - 2 - 2 = 0$ in $\mathbb{Z}_3$

Counting flows

- number of nowhere-zero G-flows of Γ is $\phi(\Gamma; |G|)$ (flow polynomial evaluated a $|G|$)
- $\phi(\Gamma) = \begin{cases} \phi(\Gamma/e) - \phi(\Gamma\backslash e) & \text{for } e \text{ not a bridge} \\ 0 & \text{for } e \text{ a bridge.} \end{cases}$

Counting flows

$\Gamma = (V, E)$, $k(\Gamma)$ conn. cpts.
rank $r(\Gamma) = |V| - k(\Gamma)$, nullity $n(\Gamma) = |E| - r(\Gamma)$
$f_G(\Gamma)$ number of G-flows (zero allowed).

1. $f_G(\Gamma) = f_G(\Gamma/e)$ for non-loop edge e

Counting flows

$\Gamma = (V, E)$, $k(\Gamma)$ conn. cpts.
rank $r(\Gamma) = |V| - k(\Gamma)$, nullity $n(\Gamma) = |E| - r(\Gamma)$
$f_G(\Gamma)$ number of G-flows (zero allowed).

1. $f_G(\Gamma) = f_G(\Gamma/e)$ for non-loop edge e
2. $f_G(\Gamma) = |G|^n$ when Γ consists of n loops alone.

Counting flows

$\Gamma = (V, E)$, $k(\Gamma)$ conn. cpts.
rank $r(\Gamma) = |V| - k(\Gamma)$, nullity $n(\Gamma) = |E| - r(\Gamma)$
$f_G(\Gamma)$ number of G-flows (zero allowed).

1. $f_G(\Gamma) = f_G(\Gamma/e)$ for non-loop edge e
2. $f_G(\Gamma) = |G|^n$ when Γ consists of n loops alone.
3. By contracting edges in spanning tree (maximal spanning forest) of Γ, leaving $n(\Gamma)$ loops,
   $f_G(\Gamma) = |G|^{n(\Gamma)}$ for any given Γ

Counting flows

$\Gamma = (V, E)$, $k(\Gamma)$ conn. cpts.
rank $r(\Gamma) = |V| - k(\Gamma)$, nullity $n(\Gamma) = |E| - r(\Gamma)$
$f_G(\Gamma)$ number of G-flows (zero allowed).

1. $f_G(\Gamma) = f_G(\Gamma/e)$ for non-loop edge e
2. $f_G(\Gamma) = |G|^n$ when Γ consists of n loops alone.
3. By contracting edges in spanning tree (maximal spanning forest) of Γ, leaving $n(\Gamma)$ loops,
   $f_G(\Gamma) = |G|^{n(\Gamma)}$ for any given Γ
4. G-flows of $\Gamma \backslash A^c \leftrightarrow G$-flows of Γ with support $\subseteq A$

Counting flows

$\Gamma = (V, E)$, $k(\Gamma)$ conn. cpts.
rank $r(\Gamma) = |V| - k(\Gamma)$, nullity $n(\Gamma) = |E| - r(\Gamma)$
$f_G(\Gamma)$ number of G-flows (zero allowed).

1. $f_G(\Gamma) = f_G(\Gamma/e)$ for non-loop edge e
2. $f_G(\Gamma) = |G|^n$ when Γ consists of n loops alone.
3. By contracting edges in spanning tree (maximal spanning forest) of Γ, leaving $n(\Gamma)$ loops,
   $f_G(\Gamma) = |G|^{n(\Gamma)}$ for any given Γ
4. G-flows of $\Gamma \backslash A^c \leftrightarrow G$-flows of Γ with support $\subseteq A$
5. By inclusion-exclusion, $\phi(\Gamma; |G|) = \sum_{A \subseteq E} (-1)^{|A^c|} |G|^{n(\Gamma \backslash A^c)}$

The Tutte polynomial

- $\phi(\Gamma;|G|) = \qquad\qquad \sum_{A\subseteq E}(-1)^{|A^c|}|G|^{n(\Gamma\backslash A^c)}$
- $\chi(\Gamma;|G|) = |G|^{k(\Gamma)}\sum_{A\subseteq E}(-1)^{|A|}|G|^{r(\Gamma/A)}$, in which $r(\Gamma/A) = r(\Gamma) - r(\Gamma\backslash A^c)$.

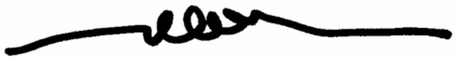
The Tutte polynomial

- $\phi(\Gamma;|G|) = \sum_{A\subseteq E}(-1)^{|A^c|}|G|^{n(\Gamma\backslash A^c)}$
- $\chi(\Gamma;|G|) = |G|^{k(\Gamma)}\sum_{A\subseteq E}(-1)^{|A|}|G|^{r(\Gamma/A)}$, in which $r(\Gamma/A) = r(\Gamma) - r(\Gamma\backslash A^c)$.

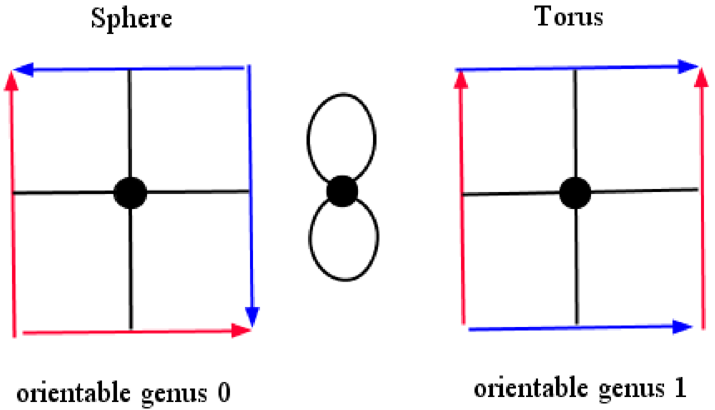
Dichromate/ Tutte polynomial:

$$T(\Gamma;x,y) = \sum_{A\subseteq E}(x-1)^{r(\Gamma)-r(\Gamma\backslash A^c)}(y-1)^{n(\Gamma\backslash A^c)}$$

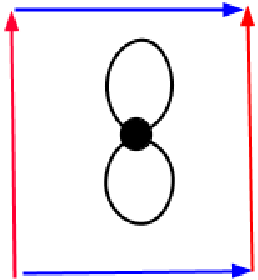
- $\chi(\Gamma;x) = (-1)^{r(\Gamma)}x^{k(\Gamma)}T(\Gamma;1-x,0)$
- $\phi(\Gamma;y) = (-1)^{n(\Gamma)}T(\Gamma;0,1-y)$
- $T(\Gamma^*;x,y) = T(\Gamma;y,x)$ for planar Γ

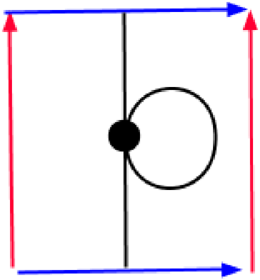


Maps



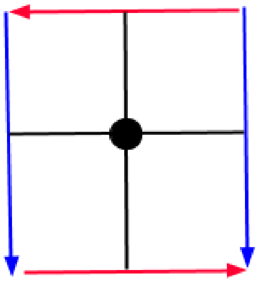
Torus





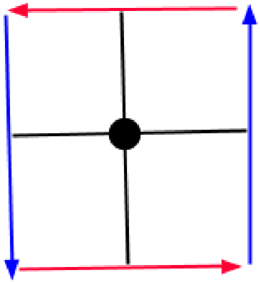
not 2-cell embeddings

Klein bottle

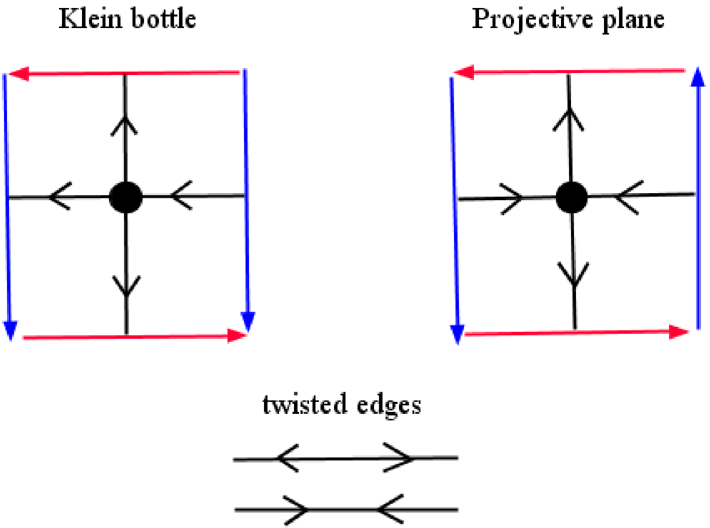


nonorientable genus 2

Projective plane



nonorientable genus 1


Klein bottle
Projective plane
twisted edges

Local tensions of maps

Map $M = (V, E, F)$ with fixed bidirection of edges.
untwisted edges: $\rightarrow\rightarrow$ or opposite $\leftarrow\leftarrow$
twisted edges: $\leftarrow\rightarrow$ or opposite $\rightarrow\leftarrow$
Finite group G (written multiplicatively).

Traversing a closed walk follows consistent bidirections on consecutive traversed edges (e.g. $\rightarrow\rightarrow$ followed by $\rightarrow\leftarrow$)

Local tensions of maps

Map $M = (V, E, F)$ with fixed bidirection of edges.
untwisted edges: $\rightarrow\rightarrow$ or opposite $\leftarrow\leftarrow$
twisted edges: $\leftarrow\rightarrow$ or opposite $\rightarrow\leftarrow$
Finite group G (written multiplicatively).

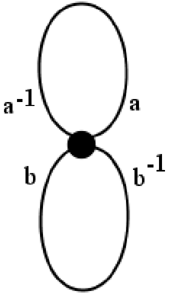
Traversing a closed walk follows consistent bidirections on consecutive traversed edges (e.g. $\rightarrow\rightarrow$ followed by $\rightarrow\leftarrow$)

- Defining property of local G-tension: net product of values around any contractible closed walk is identity (invert values when traversing by bidirection opposite to fixed one)
- Suffices to check irreducible contractible closed walks (faces)
- Nowhere-identity G local tensions of M correspond to certain proper colourings of covering graph of M [Litjens, Sevenster 2017].

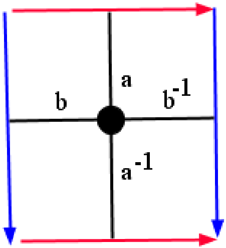
Local flows of maps

- Kirchhoff rule at each vertex (cyclic order given by vertex rotation of embedding)
- Local G-flows of $M \leftrightarrow$ local G-tensions of dual M^* (vertices of $M \leftrightarrow$ faces of M^*)

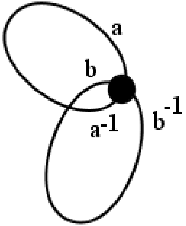
Bouquet of two loops, orientably embedded.



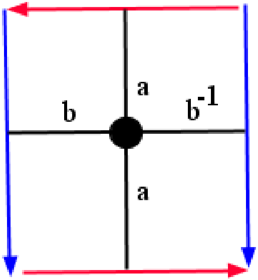
$aa^{-1}bb^{-1} = 1$



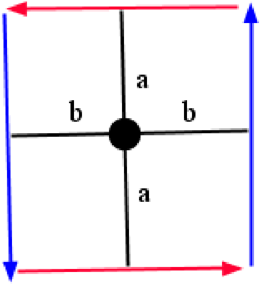
$aba^{-1}b^{-1} = 1$



Bouquet of two loops, nonorientably embedded.



$abab^{-1} = 1$



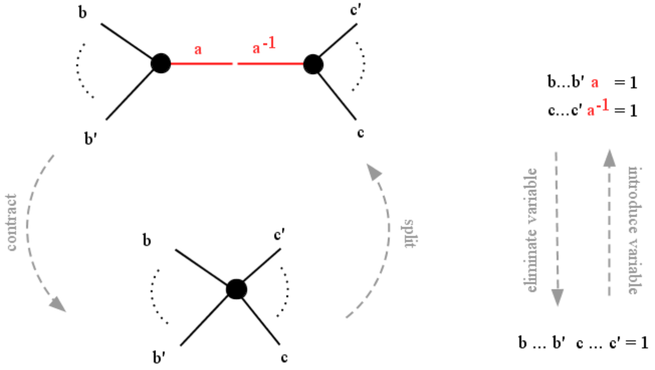
$abab=1$

Counting local flows

$f_G(M)$ number of local G-flows (identity allowed).

1A $f_G(M) = f_G(M/e)$ for non-loop untwisted edge e

- vertex-splitting, "inverse operation" to contraction



1B Sequence of vertex splittings/contractions reduces M to a bouquet M_g in standard form (cf. classification theorem for closed surfaces) of same genus g as M.

1B Sequence of vertex splittings/contractions reduces M to a bouquet M_g in standard form (cf. classification theorem for closed surfaces) of same genus g as M.

2 Finding $f_G(M)$ reduced to known enumeration of homomorphisms from fundamental group of a surface of genus g to a finite group G:

$$f_G(M) = \begin{cases} |G|^{n-1} \sum_{\rho\in\hat{G}} F(\rho)^g d_\rho^{2-g} & M \text{ non-orientable,} \\ |G|^{n-1} \sum_{\rho\in\hat{G}} d_\rho^{2-2g} & M \text{ orientable} \end{cases}$$

when M is a standard bouquet of n loops of genus g, where d_ρ is the dimension of irreducible representation ρ of G, and

$$F(\rho) = |G|^{-1} \sum_{x\in G} \chi_\rho(x^2) \in \{-1, 0, 1\}$$

is the Frobenius indicator.

3 By contracting edges in spanning tree of M, leaving $n(M) = |E| - |V| + k(M)$ loops,

$$f_G(M) = \begin{cases} |G|^{n(M)-1} \sum_{\rho \in \hat{G}} F(\rho)^g(M) d_\rho^{2-g(M)} & M \text{ non-orientable,} \\ |G|^{n(M)-1} \sum_{\rho \in \hat{G}} d_\rho^{2-2g(M)} & M \text{ orientable} \end{cases}$$

3 By contracting edges in spanning tree of M, leaving $n(M) = |E| - |V| + k(M)$ loops,

$$f_G(M) = \begin{cases} |G|^{n(M)-1} \sum_{\rho\in\hat{G}} F(\rho)^{g}(M) d_\rho^{2-g(M)} & M \text{ non-orientable,} \\ |G|^{n(M)-1} \sum_{\rho\in\hat{G}} d_\rho^{2-2g(M)} & M \text{ orientable} \end{cases}$$

4 G-flows of $M\backslash A^c \leftrightarrow G$-flows of M with support $\subseteq A$

3 By contracting edges in spanning tree of M, leaving $n(M) = |E| - |V| + k(M)$ loops,

$$f_G(M) = \begin{cases} |G|^{n(M)-1} \sum_{\rho\in\hat{G}} F(\rho)^g(M) d_\rho^{2-g(M)} & M \text{ non-orientable,} \\ |G|^{n(M)-1} \sum_{\rho\in\hat{G}} d_\rho^{2-2g(M)} & M \text{ orientable} \end{cases}$$

4 G-flows of $M \backslash A^c \leftrightarrow G$-flows of M with support $\subseteq A$

5 Inclusion-exclusion gives $\#\{$nowhere-identity G-flows of $M\} =$

$$\sum_{A\subseteq E} (-1)^{|A^c|} |G|^{|A|-|V|} \times$$

$$\prod_{\substack{\text{orient.}\\ \text{conn. cpts}\\ M_i \text{ of } M\backslash A^c}} \sum_{\rho\in\widehat{G}} d_\rho^{2-2g(M_i)} \prod_{\substack{\text{non-orient.}\\ \text{conn. cpts}\\ M_j \text{ of } M\backslash A^c}} \sum_{\rho\in\widehat{G}} F(\rho)^{g(M_j)} d_\rho^{2-g(M_j)}.$$

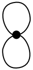
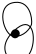
Definition

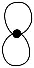
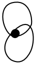
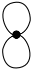
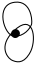
Let $\mathbf{x} = (x; \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ and $\mathbf{y} = (y; \ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$ be infinite sequences of indeterminates.

The surface Tutte polynomial of a map $M = (V, E, F)$ is the multivariate polynomial $\mathcal{T}(M; \mathbf{x}, \mathbf{y}) :=$

$$\sum_{A \subseteq E} x^{n^*(M/A)} y^{n(M \backslash A^c)} \prod_{\substack{\text{conn. cpts} \\ M_i \text{ of } M/A}} x_{\bar{g}(M_i)} \prod_{\substack{\text{conn. cpts} \\ M_j \text{ of } M \backslash A^c}} y_{\bar{g}(M_j)},$$

where $A^c = E \backslash A$ for $A \subseteq E$, $n^*(M) = |E| - |F| + k(M) = n(M^*)$, and $\bar{g}(M)$ is the signed genus of M.

$M=$  (edges b, a) $\qquad M=$  (edges b, a)

$A=$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
M/A								
$x^{n^*(M/A)}$	1	1	1	1	x^2	x	x	1
$\prod x_{g(M_i)}$	x_0	x_0^2	x_0^2	x_0^3	x_1	x_0	x_0	x_0
$M\backslash A^c$								
$y^{n(M\backslash A^c)}$	1	y	y	y^2	1	y	y	y^2
$\prod y_{g(M_j)}$	y_0	y_0	y_0	y_0	y_0	y_0	y_0	y_1
	$\mathcal{T}($$;\mathbf{x},\mathbf{y}) = x_0y_0 + 2yx_0^2y_0 + y^2x_0^3y_0$				$\mathcal{T}($$;\mathbf{x},\mathbf{y}) = x^2x_1y_0 + 2xyx_0y_0 + y^2x_0y_1$			

Theorem (G. Krajewski, Regts, Vena 2018; G., Litjens, Regts, Vena, 2018+)

Let G be a finite group with irreducible representations ρ of dimension d_ρ. Then

$$\#\{\text{nowhere-identity local } G\text{-flows of } M\} = (-1)^{|E|-|V|}\mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

- $x = 1, y = -|G|$;
- $x_g = 1$ *and* $y_g = -|G|^{-1}\sum_{\rho\in\widehat{G}} d_\rho^{2-2g}$ *for* $g \geq 0$; *and*
- $x_{-g} = 1$ *and* $y_{-g} = -|G|^{-1}\sum_{\rho\in\widehat{G}} F(\rho)^g d_\rho^{2-g}$ *for* $g \geq 1$.

Theorem (G. Krajewski, Regts, Vena 2018; G., Litjens, Regts, Vena, 2018+)

Let G be a finite group with irreducible representations ρ of dimension d_ρ. Then

$$\#\{\textit{nowhere-identity local } G\textit{-flows of } M\} = (-1)^{|E|-|V|}\mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

- $x = 1, y = -|G|$;
- $x_g = 1$ *and* $y_g = -|G|^{-1}\sum_{\rho\in\widehat{G}} d_\rho^{2-2g}$ *for* $g \geq 0$; *and*
- $x_{-g} = 1$ *and* $y_{-g} = -|G|^{-1}\sum_{\rho\in\widehat{G}} F(\rho)^g d_\rho^{2-g}$ *for* $g \geq 1$.
- *(For $g \geq 0$, y_g is $|G|^{-2g} f_G(M_g)$; for $g \geq 1$, y_{-g} is $|G|^g f_G(M_-g)$, where $M_{\pm g}$ is a bouquet of signed genus $\pm g$.)*

Theorem (G. Krajewski, Regts, Vena 2018; G., Litjens, Regts, Vena, 2018+)

Let G be a finite group with irreducible representations ρ of dimension d_ρ. Then

$$\#\{\text{nowhere-identity local } G\text{-flows of } M\} = (-1)^{|E|-|V|}\mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

- $x = 1, y = -|G|$;
- $x_g = 1$ and $y_g = -|G|^{-1}\sum_{\rho\in\widehat{G}} d_\rho^{2-2g}$ for $g \geq 0$; and
- $x_{-g} = 1$ and $y_{-g} = -|G|^{-1}\sum_{\rho\in\widehat{G}} F(\rho)^g d_\rho^{2-g}$ for $g \geq 1$.
- (For $g \geq 0$, y_g is $|G|^{-2g} f_G(M_g)$; for $g \geq 1$, y_{-g} is $|G|^g f_G(M_-g)$, where $M_{\pm g}$ is a bouquet of signed genus $\pm g$.)

$$\#\{\text{nowhere-identity local } G\text{-tensions}\} = (-1)^{|E|-|F|}\mathcal{T}(M;\mathbf{y},\mathbf{x}).$$

Other specializations

- Tutte polynomial $T(\Gamma; X, Y)$: $x = 1 = x_g, y = Y - 1$ and $y_g = X - 1$ (for arbitrary embedding of Γ)
- Embedding a signed graph $\Sigma = (\Gamma, \sigma)$ in a surface (embedded cycles orientation-preserving precisely when balanced) and taking $x = 1 = x_g, y = Y - 1$, and $y_g = X - 1$ if $g \geq 0$, $y_g = (X - 1)(Z - 1)/(Y - 1)$ if $g \leq -1$ gives a new "signed Tutte polynomial" $T_\Sigma(X, Y, Z) =$

$$\sum_{A \subseteq E} (X-1)^{k(\Sigma \backslash A^c) - k(\Sigma)} (Y-1)^{|A| - |V| + k_b(\Sigma \backslash A^c)} (Z-1)^{k(\Sigma \backslash A^c) - k_b(\Sigma \backslash A^c)}$$

where k_b denotes the number of balanced components.

[G., Litjens, Regts, Vena, 2018+] The signed Tutte polynomial $T_\Sigma(X, Y, Z)$ contains as evaluations the number of

- proper $\{0, \pm 1, \ldots, \pm n\}$-colourings of Σ, at $(-2n, 0, \frac{2n}{2n+1})$,

  and, for finite abelian group G,

- the number of nowhere-zero G-tensions of Σ, at $(1 - |G|, 0, \frac{|G|-1}{m})$
- the number of nowhere-zero G-flows of Σ, at $(0, 1 - |G|, 1 - 2^d)$

where $|G| = 2^d m$, with 2^d the size of G's 2-torsion subgroup.

(Tensions are required to have net sum of values zero around only balanced (positive) irreducible closed walks of Σ.)


The End